Time parameters and Lorentz transformations of relativistic stochastic processes

Jörn Dunkel,[1,*] Peter Hänggi,[1] and Stefan Weber[2]
[1] Institut für Physik, Universität Augsburg, Universitätsstraße 1, Augsburg, Germany
[2] School of Operations Research and Information Engineering,
279 Rhodes Hall, Cornell University, Ithaca, NY 14853, USA
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Rules for the transformation of time parameters in relativistic Langevin equations are derived and discussed. In particular, it is shown that, if a coordinate-time parameterized process approaches the relativistic Jüttner-Maxwell distribution, the associated proper-time parameterized process converges to a modified momentum distribution, differing by a factor proportional to the inverse energy.

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Stochastic processes (SPes) present an ubiquitous tool for modelling complex phenomena in physics [1–3], biology [4, 5], or economics and finance [6–9]. Stochastic concepts provide a promising alternative to deterministic models whenever the underlying microscopic dynamics of a relevant observable is not known exactly but plausible assumptions about the underlying statistics can be made. A specific area where the formulation of consistent microscopic interaction models becomes difficult [10–12] concerns classical relativistic many-particle systems. Accordingly, SPes provide a useful phenomenological approach to describing, e.g., the interaction of a relativistic particle with a fluctuating environment [13–17]. Applications of stochastic concepts to relativistic problems include thermalization processes in quark-gluon plasmas, as produced in relativistic heavy ion colliders [18–21], or complex high-energy processes in astrophysics [22–25].

While these applications illustrate the practical relevance of relativistic SPes, there still exist severe conceptual issues which need clarification from a theoretical point of view. Among these is the choice of the time-parameter that quantifies the evolution of a relativistic SP [26]. This problem does not arise within a nonrelativistic framework, since the Newtonian physics postulates the existence of a universal time which is the same for any inertial observer; thus, it is natural to formulate nonrelativistic SPes by making reference to this universal time. By contrast, in special relativity [27, 28] the notion of time becomes frame-dependent, and it is necessary to carefully distinguish between different time parameters when constructing relativistic SPes. For example, if the random motion of a relativistic particle is described in a t-parameterized form, where t is the time coordinate of some fixed inertial system Σ, then one may wonder if/how this process can be re-expressed in terms of the particle's proper-time τ, and *vice versa*. Another closely related question [17] concerns the problem of how a certain SP appears to a moving observer, i.e.: How does a SP behave under a Lorentz transformation?

The present paper aims at clarifying the above questions for a broad class of relativistic SPes governed by relativistic Langevin equations [13–17]. First, we will discuss a heuristic approach that suffices for most practical calculations and clarifies the basic ideas. Subsequently, these heuristic arguments will be substantiated with a mathematically rigorous foundation by applying theorems for the time-change of (local) martingale processes [29]. The main results can be summarized as follows: If a relativistic Langevin-Itô process has been specified in the inertial frame Σ and is parameterized by the associated Σ-coordinate time t, then this process can be reparameterized by its proper-time τ and the resulting process is again of the Langevin-Itô type. Furthermore, the process can be Lorentz transformed to a moving frame Σ', yielding a Langevin-Itô process that is parameterized by the Σ'-time t'. In other words, similar to the case of purely deterministic relativistic equations of motions, one can choose freely between different time parameterizations in order to characterize these relativistic SPes – but the noise part needs to be transformed differently than the deterministic part.

Notation.– We adopt the metric convention $(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, \ldots, 1)$ and units such that the speed of light $c = 1$. Contra-variant space-time and energy-momentum four-vectors are denoted by $(x^\alpha) = (x^0, x^i) = (x^0, \boldsymbol{x}) = (t, \boldsymbol{x})$ and $(p^\alpha) = (p^0, p^i) = (p^0, \boldsymbol{p})$, respectively, with Greek indices $\alpha = 0, 1, \ldots, d$ and Latin indices $i = 1, \ldots, d$, where d is the number of space dimensions. Einstein's summation convention is applied throughout.

Relativistic Langevin equations.– As a starting point, we consider the t-parameterized random motion of a relativistic particle (rest mass M) in the inertial lab frame Σ. The lab frame is defined by the property that the thermalized background medium (heat bath) causing the stochastic motion of the particle is at rest in Σ (on average). We assume that the particle's trajectory $(\boldsymbol{X}(t), \boldsymbol{P}(t)) = (X^i(t), P^i(t))$ in Σ is governed by a stochastic differential equation (SDE) of the form [13–17]

$$\mathrm{d}X^\alpha(t) = (P^\alpha/P^0)\,\mathrm{d}t, \tag{1a}$$

$$\mathrm{d}P^i(t) = A^i\,\mathrm{d}t + C^i{}_j\mathrm{d}B^j(t). \tag{1b}$$

Here, $\mathrm{d}X^0(t) = \mathrm{d}t$ and $\mathrm{d}X^i(t) := X^i(t + \mathrm{d}t) - X^i(t)$ denote the time and position increments, $\mathrm{d}P^i(t) :=$

$P^i(t+\mathrm{d}t) - P^i(t)$ the momentum change. $P^0(t) := (M^2 + \boldsymbol{P}^2)^{1/2}$ is the relativistic energy, and $V^i(t) := \mathrm{d}X^i/\mathrm{d}t = P^i/P^0$ are the velocity components in Σ. In general, the functions A^i and $C^i{}_j$ may depend on the time, position and momentum coordinates of the particle. The random driving process $\boldsymbol{B}(t) = (B^j(t))$ is taken to be a d-dimensional t-parameterized standard Wiener process (WP) [29–31], i.e., $\boldsymbol{B}(t)$ has continuous paths, for $s > t$ the increments are normally distributed,

$$\mathcal{P}\{\boldsymbol{B}(s) - \boldsymbol{B}(t) \in [\boldsymbol{u}, \boldsymbol{u} + \mathrm{d}\boldsymbol{u}]\} = \frac{e^{-|\boldsymbol{u}|^2/[2\,(s-t)]}}{[2\pi\,(s-t)]^{d/2}}\,\mathrm{d}^d u, \quad (2)$$

and independent for non-overlapping time intervals [40].

Upon naively dividing Eq. (1b) by $\mathrm{d}t$, we see that A^i can be interpreted as a deterministic force component, while $C^i{}_j \mathrm{d}B^j(t)/\mathrm{d}t$ represents random 'noise'. However, for the Wiener process the derivatives $\mathrm{d}B^j(t)/\mathrm{d}t$ are not well-defined mathematically so the differential representation (1) is in fact short hand for a stochastic integral equation [29, 31] with $C^i{}_j \mathrm{d}B^j$ signifying an infinitesimal increment of the Itô integral [32, 33]. Like a deterministic integral, stochastic integrals can be approximated by Riemann-Stieltjes sums but the coefficient functions need to be evaluated at the *left* end point t of any time interval $[t, t+\mathrm{d}t]$ in the Itô discretization [41]. In contrast to other discretization rules [1, 29, 31, 34, 35], the Itô discretization implies that the mean value of the noise vanishes, i.e., $\langle C^i{}_j \mathrm{d}B^j(t)\rangle = 0$ with $\langle\,\cdot\,\rangle$ indicating an average over all realizations of the Wiener process $\boldsymbol{B}(t)$. In other words, Itô integrals with respect to $\boldsymbol{B}(t)$ are (local) martingales [29]. Upon applying Itô's formula [29, 31] to the mass-shell condition $P^0(t) = (M^2 + \boldsymbol{P}^2)^{1/2}$, one can derive from Eq. (1b) the following equation for the relativistic energy:

$$\begin{aligned} \mathrm{d}P^0(t) &= A^0\,\mathrm{d}t + C^0{}_r \mathrm{d}B^r(t), \qquad (3)\\ A^0 &:= \frac{A_i P^i}{P^0} + \frac{D_{ij}}{2}\left[\frac{\delta^{ij}}{P^0} - \frac{P^i P^j}{(P^0)^3}\right], \quad C^0{}_j := \frac{P^i C_{ij}}{P^0}, \end{aligned}$$

where $A_i := A^i$, $D_{ij} := D^{ij} = \sum_r C^i_r C^j_r$ and $C_{ir} := C^i{}_r$.

Equations (1) define a straightforward relativistic generalization [13–15] of the classical Ornstein-Uhlenbeck process [36], representing a standard model of Brownian motion theory [42]. The structure of Eq. (1a) ensures that the velocity remains bounded, $|\boldsymbol{V}| < 1$, even if the momentum $\boldsymbol{P}$ were to become infinitely large. When studying SDEs of the type (1), one is typically interested in the probability $f(t, \boldsymbol{x}, \boldsymbol{p})\,\mathrm{d}^d x \mathrm{d}^d p$ of finding the particle at time t in the infinitesimal phase space interval $[\boldsymbol{x}, \boldsymbol{x}+\mathrm{d}\boldsymbol{x}] \times [\boldsymbol{p}, \boldsymbol{p}+\mathrm{d}\boldsymbol{p}]$. Given Eqs. (1), the non-negative, normalized probability density $f(t, \boldsymbol{x}, \boldsymbol{p})$ is governed by the Fokker-Planck equation (FPE)

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{p^0}\frac{\partial}{\partial x^i}\right) f = \frac{\partial}{\partial p^i}\left[-A^i f + \frac{1}{2}\frac{\partial}{\partial p^k}\left(D^{ik} f\right)\right], \quad (4)$$

where $p^0 = (M^2 + \boldsymbol{p}^2)^{1/2}$. Deterministic initial data $\boldsymbol{X}(0) = \boldsymbol{x}_0$ and $\boldsymbol{P}(0) = \boldsymbol{p}_0$ translates into the localized initial condition $f(0, \boldsymbol{x}, \boldsymbol{p}) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\,\delta(\boldsymbol{p} - \boldsymbol{p}_0)$. Physical constraints on the coefficients $A^i(t, \boldsymbol{x}, \boldsymbol{p})$ and $C^i{}_r(t, \boldsymbol{x}, \boldsymbol{p})$ may arise from symmetries and/or thermodynamic considerations. For example, neglecting additional external force fields and considering a heat bath that is stationary, isotropic and position independent in Σ, one is led to the ansatz

$$A^i = -\alpha(p^0)\,p^i\,, \qquad C^i{}_j = [2D(p^0)]^{1/2}\,\delta^i{}_j. \quad (5a)$$

where the friction and noise coefficients α and D depend on the energy p^0 only. Moreover, if the stationary momentum distribution is expected to be a thermal Jüttner function [37, 38], i.e., if $f_\infty := \lim_{t\to\infty} f \propto \exp(-\beta p^0)$ in Σ, then α and D must satisfy the fluctuation-dissipation condition [13]

$$0 \equiv \alpha(p^0)\,p^0 + \mathrm{d}D(p^0)/\mathrm{d}p^0 - \beta D(p^0). \quad (5b)$$

In this case, one still has the freedom to adapt one of the two functions α or D.

In the remainder, we shall discuss how the process (1) can be reparameterized in terms of its proper-time τ, and how it transforms under the proper Lorentz group [28].

Proper-time parameterization.– The proper-time differential $\mathrm{d}\tau(t) = (1 - \boldsymbol{V}^2)^{1/2}\mathrm{d}t$ may be expressed as

$$\mathrm{d}\tau(t) = (M/P^0)\,\mathrm{d}t. \quad (6a)$$

The inverse of the function τ is denoted by $\hat{X}^0(\tau) = t(\tau)$ and represents the time coordinate of the particle in the inertial frame Σ, parameterized by the proper time τ. Our goal is to find SDEs for the reparameterized processes $\hat{X}^\alpha(\tau) := X^\alpha(t(\tau))$ and $\hat{P}^\alpha(\tau) = P^\alpha(t(\tau))$ in Σ. The heuristic derivation is based on the relation

$$\mathrm{d}B^j(t) \simeq \sqrt{\mathrm{d}t} = \left(\frac{\hat{P}^0}{M}\right)^{1/2}\sqrt{\mathrm{d}\tau} \simeq \left(\frac{\hat{P}^0}{M}\right)^{1/2}\mathrm{d}\hat{B}^j(\tau), \quad (6b)$$

where $\hat{B}^j(\tau)$ is a standard Wiener process with time-parameter τ. The rigorous justification of Eq. (6b) is given below. Inserting Eqs. (6) in Eqs. (1) one finds

$$\begin{aligned} \mathrm{d}\hat{X}^\alpha(\tau) &= (\hat{P}^\alpha/M)\,\mathrm{d}\tau, \qquad (7a)\\ \mathrm{d}\hat{P}^i(\tau) &= \hat{A}^i\,\mathrm{d}\tau + \hat{C}^i{}_j \mathrm{d}\hat{B}^j(\tau), \qquad (7b) \end{aligned}$$

where $\hat{A}^i := (\hat{P}^0/M)\,A^i(\hat{X}^0, \hat{\boldsymbol{X}}, \hat{\boldsymbol{P}})$ and $\hat{C}^i{}_j := (\hat{P}^0/M)^{1/2}\,C^i{}_j(\hat{X}^0, \hat{\boldsymbol{X}}, \hat{\boldsymbol{P}})$. The FPE for the associated probability density $\hat{f}(\tau, x^0, \boldsymbol{x}, \boldsymbol{p})$ reads

$$\left(\frac{\partial}{\partial \tau} + \frac{p^\alpha}{M}\frac{\partial}{\partial x^\alpha}\right)\hat{f} = \frac{\partial}{\partial p^i}\left[-\hat{A}^i \hat{f} + \frac{1}{2}\frac{\partial}{\partial p^k}\left(\hat{D}^{ik}\hat{f}\right)\right] \quad (8)$$

where now $\hat{D}^{ik} := \sum_r \hat{C}^i_r \hat{C}^k_r$. We note that $\hat{f}(\tau, x^0, \boldsymbol{x}, \boldsymbol{p})\,\mathrm{d}x^0\mathrm{d}^d x \mathrm{d}^d p$ gives probability of finding the particle at proper-time τ in the interval $[t, t+\mathrm{d}t] \times [\boldsymbol{x}, \boldsymbol{x} + \mathrm{d}\boldsymbol{x}] \times [\boldsymbol{p}, \boldsymbol{p} + \mathrm{d}\boldsymbol{p}]$ in the inertial frame Σ.

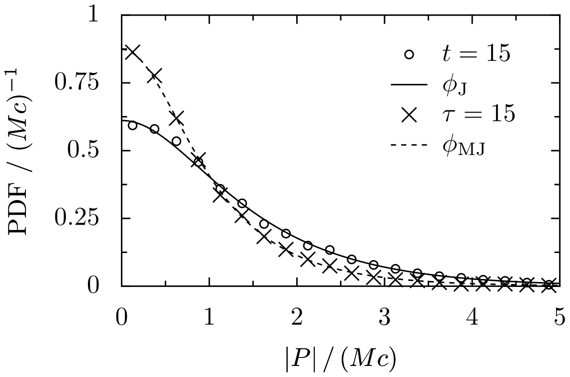

FIG. 1: 'Stationary' probability density function (PDF) of the absolute momentum $|P|$ measured at time $t = 15$ ($\times$) and $\tau = 15$ ($\circ$) from 10000 sample trajectories of the one-dimensional ($d = 1$) relativistic Ornstein-Uhlenbeck process [13], corresponding to coefficients $D(p^0) = const$ and $\alpha(p^0) = \beta D/p^0$ in Eqs. (1) and (5). Simulation parameters: $\mathrm{d}t = 0.001$, $M = c = \beta = D = 1$.

Remarkably, if the coefficient functions satisfy the constraints (5) – so that the stationary solution f_∞ of Eq. (4) is a Jüttner function $\phi_{\rm J}(\boldsymbol{p}) = \mathcal{Z}^{-1}\exp(-\beta p^0)$ – then the stationary solution $\hat f_\infty$ of the corresponding proper-time FPE (8) is given by a modified Jüttner function $\phi_{\rm MJ}(\boldsymbol{p}) = \hat{\mathcal{Z}}^{-1}\exp(-\beta p^0)/p^0$. The latter can be derived from a relative entropy principle, using a Lorentz invariant reference measure in momentum space [39]. Physically, the difference between f_∞ and $\hat f_\infty$ is due to the fact that measurements at $t = const$ and $\tau = const$ are non-equivalent even if $\tau, t \to \infty$. This can also be confirmed by direct numerical simulation of Eqs. (1), see Fig. 1.

Having discussed the proper-time reparameterization, we next show that a similar reasoning can be applied to transform the SDEs (1) to a moving frame Σ' [17].

Lorentz transformations.– Neglecting time-reversals, we consider a proper Lorentz transformation [28] from the lab frame Σ to Σ', mediated by a constant matrix $\Lambda^\nu{}_\mu$ with $\Lambda^0{}_0 > 0$, that leaves the metric tensor $\eta_{\alpha\beta}$ invariant. We proceed in two steps: First we define

$$Y'^\nu(t) := \Lambda^\nu{}_\mu X^\mu(t)\;, \qquad G'^\nu(t) := \Lambda^\nu{}_\mu P^\mu(t).$$

Then we replace t by the coordinate time t' of Σ' to obtain processes $X'^\alpha(t') = Y'^\alpha(t(t'))$ and $P'^\alpha(t') = G'^\alpha(t(t'))$. Note that $\mathrm{d}t'(t) = \mathrm{d}Y'^0(t) = \Lambda^0{}_\mu \mathrm{d}X^\mu(t)$, and, hence,

$$\mathrm{d}t'(t) = \frac{\Lambda^0{}_\mu P^\mu}{P^0}\,\mathrm{d}t = \frac{G'^0}{P^0}\,\mathrm{d}t = \frac{P'^0(t'(t))}{(\Lambda^{-1})^0{}_\mu P'^\mu(t'(t))}\,\mathrm{d}t, \quad (9)$$

where Λ^{-1} is the inverse Lorentz transformation. Thus, a similar heuristics as in Eq. (6b) gives

$$\mathrm{d}B^j(t) \simeq \sqrt{\mathrm{d}t} = \left(\frac{P^0}{P'^0}\right)^{1/2}\sqrt{\mathrm{d}t'} \simeq \left[\frac{(\Lambda^{-1})^0{}_\mu P'^\mu}{P'^0}\right]^{1/2}\mathrm{d}B'^j(t'),$$

where $B'^j(t')$ is a Wiener process with time t'. Furthermore, defining primed coefficient functions in Σ' by

$$\begin{aligned} A'^i(x'^0,\boldsymbol{x}',\boldsymbol{p}') &:= [(\Lambda^{-1})^0{}_\mu p'^\mu/p'^0]\;\times \\ &\Lambda^i{}_\nu\, A^\nu\left((\Lambda^{-1})^0{}_\mu x'^\mu, (\Lambda^{-1})^i{}_\mu x'^\mu, (\Lambda^{-1})^i{}_\mu p'^\mu\right), \\ C'^i{}_j(x'^0,\boldsymbol{x}',\boldsymbol{p}') &:= [(\Lambda^{-1})^0{}_\mu p'^\mu/p'^0]^{1/2}\times \\ &\Lambda^i{}_\nu\, C^\nu{}_j\left((\Lambda^{-1})^0{}_\mu x'^\mu, (\Lambda^{-1})^i{}_\mu x'^\mu, (\Lambda^{-1})^i{}_\mu p'^\mu\right), \end{aligned}$$

the particle's trajectory $(\boldsymbol{X}'(t'), \boldsymbol{P}'(t'))$ in Σ' is again governed by a SDE of the standard form

$$\mathrm{d}X'^\alpha(t') \;=\; (P'^\alpha/P'^0)\,\mathrm{d}t', \quad (11a)$$
$$\mathrm{d}P'^i(t') \;=\; A'^i\,\mathrm{d}t' + C'^i{}_j\,\mathrm{d}B'^j(t'). \quad (11b)$$

Rigorous justification.– We will now rigorously derive the transformations of SDEs under time changes and thereby show that the heuristic transformations leading to Eqs. (7) and (11) are justified; i.e., we are interested in a time-change $t \mapsto \breve t$ of a generic SDE

$$\mathrm{d}Y(t) = E\,\mathrm{d}t + F_j\,\mathrm{d}B^j(t), \quad (12a)$$

where E and F_j will typically be smooth functions of the state-variables $(Y,\ldots)$ [43], and $\boldsymbol{B}(t) = (B^j(t))$ is a d-dimensional standard Wiener process [44]. We consider a time-change $t \mapsto \breve t$ specified in the form [cf. Eqs. (6a) and (9)]

$$\mathrm{d}\breve t = H\;\mathrm{d}t, \qquad \breve t(0) = 0, \quad (12b)$$

with H being a strictly positive smooth function [45] of $(Y,\ldots)$. The inverse of $\breve t(t)$ is denoted by $t(\breve t)$. We would like to show that Eq. (12a) can be rewritten as

$$\mathrm{d}\breve Y(\breve t) = \breve E\,\mathrm{d}\breve t + \breve F_j\,\mathrm{d}\breve B^j(\breve t), \quad (12c)$$

where $\breve Y(\breve t) := Y(t(\breve t))$, $\breve E(\breve t) := E(t(\breve t))/H(t(\breve t))$, $\breve F^j(\breve t) := F^j(t(\breve t))/\sqrt{H(t(\breve t))}$, and

$$\mathrm{d}\breve B^j(\breve t) := \sqrt{H}\;\mathrm{d}B^j(t) \quad (12d)$$

is a d-dimensional Wiener process with respect to the new time parameter $\breve t$ [46].

First, we need to prove that Eq. (12d) or, equivalently, $\breve B^j(\breve t) := \int_0^{t(\breve t)}\;\sqrt{H(s)}\;\mathrm{d}B^j(s)$ does indeed define a Wiener process. To this end, we note that for fixed $j \in \{1,\ldots,d\}$ the process $L^j(t) := \int_0^t \sqrt{H(s)}\,\mathrm{d}B^j(s)$ is a continuous local martingale, whose quadratic variation

$$[L^j, L^j](t) := \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\left\{L^j\left(\frac{(k+1)t}{2^n}\right) - L^j\left(\frac{kt}{2^n}\right)\right\}^2$$

is given by $[L^j, L^j](t) = \int_0^t H(s)\mathrm{d}s$ [47]. For the quadratic variation of $\breve B^j(\breve t) = L^j(t(\breve t))$ we then obtain $[\breve B^j, \breve B^j](\breve t) = [L^j, L^j](t(\breve t)) = \int_0^{t(\breve t)} H(s)\,\mathrm{d}s = \breve t$. For $i \neq j$, we have $[\breve B^j, \breve B^i](\breve t) = \int_0^{t(\breve t)} H(s)\,\mathrm{d}[B^j, B^i](s) = 0$. Thus, Lévy's Theorem [48] implies that $\breve{\boldsymbol{B}}(\breve t) = (\breve B^j(\breve t))$ is a d-dimensional standard Wiener process. Finally, using the definitions of $\breve{Y}$, $\breve{E}$, and $\breve{F}^j$, we find [49]

$$\begin{aligned}\breve{Y}(\breve{t}) &= \int_0^{t(\breve{t})} E(s)\,\mathrm{d}s + \int_0^{t(\breve{t})} F_j(s)\,\mathrm{d}B^j(s) \\ &= \int_0^{\breve{t}} \frac{E(t(\breve{s}))}{H(t(\breve{s}))} d\breve{s} + \int_0^{\breve{t}} \frac{F_j(t(\breve{s}))}{\sqrt{H(t(\breve{s}))}}\,\mathrm{d}\breve{B}^j(\breve{s}) \\ &= \int_0^{\breve{t}} \breve{E}(\breve{s})\,\mathrm{d}\breve{s} + \int_0^{\breve{t}} \breve{F}_j(\breve{s})\,\mathrm{d}\breve{B}^j(\breve{s}), \qquad (13)\end{aligned}$$

which is just the SDE (12c) written in integral notation.

Summary.– The above discussion shows how relativistic Langevin equations of the type (1) can be Lorentz transformed [17] and reparameterized within a common framework. The relativistic Langevin theory [13–17] is now as well-founded and mathematically complete as the classical theories of nonrelativistic Brownian motions and deterministic relativistic motions (which are both included as special limit cases). From a physics point of view, the most remarkable observation consists in the fact that the τ-parameterized Brownian motion converges to a modified Jüttner function [39] if the corresponding t-parameterized process converges to a Jüttner function [37]. With regard to applications [20, 21] this means that the correct form of the fluctuation-dissipation relation depends on the choice of the time-parameter in the relativistic Langevin equation.

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* Electronic address: `dunkeljo@physik.uni-augsburg.de`

[1] P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).

[2] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[3] E. Frey and K. Kroy, Ann. Phys. (Leipzig) **14**, 20 (2005).

[4] L. J. S. Allen, *Stochastic Processes with Applications to Biology* (Prentice-Hall, Upper Saddle River, NJ, 2002).

[5] D. Wilkinson, *Stochastic Modelling for Systems Biology* (Chapman & Hall/CRC, London, 2006).

[6] L. Bachelier, Ann. Sci. Ecole Norm. Sup. **17**, 21 (1900).

[7] F. Black and M. Scholes, J. Polit. Econ. **81**, 81 (1973).

[8] R. C. Merton, J. Financ. **29**, 449 (1974).

[9] S. E. Shreve, *Stochastic Calculus for Finance I & II* (Springer, Berlin, 2004).

[10] A. D. Fokker, Z. Physik A: Hadrons and Nuclei **58**, 386 (1929).

[11] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

[12] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965).

[13] F. Debbasch, K. Mallick, and J. P. Rivet, J. Stat. Phys **88**, 945 (1997).

[14] J. Dunkel and P. Hänggi, Phys. Rev. E **71**, 016124 (2005).

[15] R. Zygadło, Phys. Lett. A **345**, 323 (2005).

[16] J. Angst and J. Franchi, J. Math. Phys. **48**, 083101 (2007).

[17] C. Chevalier and F. Debbasch, J. Math. Phys. **49**, 043303 (2008).

[18] B. Svetitsky, Phys. Rev. D **37**, 2484 (1988).

[19] P. Roy, J. E. Alam, S. Sarkar, B. Sinha, and S.Raha, Nucl. Phys. A **624**, 687 (1997).

[20] H. van Hees and R. Rapp, Phys. Rev. C **71**, 034907 (2005).

[21] R. Rapp, V. Greco, and H. van Hees, Nucl. Phys. A **774**, 685 (2006).

[22] P. J. D. Mauas and D. O. Gomez, Astophys. J. **398**, 682 (1997).

[23] N. Itoh, Y. Kohyama, and S. Nozawa, Astrophys. J. **502**, 7 (1998).

[24] B. Wolfe and F. Melia, Astrophys. J. **638** (2006).

[25] P. A. Becker, T. Le, and C. D. Dermer, Astrophys. J. **647**, 539 (2006).

[26] R. Hakim, J. Math. Phys. **9**, 1805 (1968).

[27] A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905).

[28] R. U. Sexl and H. K. Urbantke, *Relativity, Groups, Particles*, Springer Physics (Springer, Wien, 2001).

[29] P. E. Protter, *Stochastic Integration and Differential Equations* (Springer-Verlag, New York, 2004).

[30] N. Wiener, J. Math. Phys. **2**, 131 (1923).

[31] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus* (Springer, New York, Berlin, 1991).

[32] K. Ito, Proc. Imp. Acad. Tokyo **20**, 519 (1944).

[33] K. Ito, Mem. Amer. Mathem. Soc. **4**, 51 (1951).

[34] D. Fisk, Trans. Amer. Math. Soc. **120**, 369 (1965).

[35] R. L. Stratonovich, SIAM J. Control **4**, 362 (1966).

[36] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).

[37] F. Jüttner, Ann. Phys. (Leipzig) **34**, 856 (1911).

[38] D. Cubero, J. Casado-Pascual, J. Dunkel, P. Talkner, and P. Hänggi, Phys. Rev. Lett. **99**, 170601 (2007).

[39] J. Dunkel, P. Talkner, and P. Hänggi, New J. Phys. **9**, 144 (2007).

[40] For simplicity, we have assumed that $\boldsymbol{B}(t)$ is d-dimensional, implying that ${C^i}_j$ is a square matrix. However, all results still hold if $\boldsymbol{B}(t)$ has a different dimension.

[41] One could also consider other discretization rules [1, 29, 31, 34, 35], but then the rules of stochastic differential calculus have to be adapted.

[42] In the nonrelativistic limit $c \to \infty$, $P^0 \to M$ in Eq. (1a).

[43] The state variables of the system are assumed to have continuous paths and need to satisfy suitable integrability conditions. More generally, $E = E(t)$ and $F_j = F_j(t)$ can be assumed to be continuous adapted processes.

[44] The Wiener process is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ that satisfies the usual hypotheses [29]. The increasing family $\mathbb{F} = (\mathcal{F}_t)$ is called a filtration. $\mathcal{F}_t$ denotes the information that will be available to an observer at time t who follows the particle.

[45] More precisely, in general $H = H(t)$ is a strictly positive, continuous adapted process such that $\mathcal{P}[\int_0^t H(s)\mathrm{d}s < \infty \forall t] = 1$ and $\mathcal{P}[\int_0^\infty H(s)\mathrm{d}s = \infty] = 1$.

[46] The information available to an observer of the particle at time $\breve{t}$ is denoted by $\mathcal{G}_{\breve{t}}$. The corresponding filtration is denoted by $\mathbb{G} = (\mathcal{G}_{\breve{t}})$; cf. Chapt. I.1 in [29]. The mathematically precise statement regarding the time-change is that $\breve{\boldsymbol{B}}(\breve{t})$ is a standard Wiener process with respect to $\mathbb{G}$.

[47] Convergence is uniform on compacts in probability; see [29] for a definition of the quadratic covariation $[L^i, L^j]_t$.

[48] See Theorem II.8.40, p. 87 in Protter [29].

[49] The second equality follows from Eqs. (12b) and (12d) by approximating the processes E and F^j by simple predictable processes, see p. 51 and Theorem II.5.21 in [29].