Solvency II, or How to Sweep the Downside Risk Under the Carpet

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Abstract

Under Solvency II the computation of capital requirements is based on value at risk (V@R). V@R is a quantile-based risk measure and neglects extreme risks in the tail. V@R belongs to the family of distortion risk measures. A serious deficiency of V@R is that firms can hide their total downside risk in corporate groups. They can largely reduce their total capital requirements via appropriate transfer agreements within a group structure consisting of sufficiently many entities and thereby circumvent capital regulation. We prove several versions of such a result for general distortion risk measures of V@R-type, explicitly construct suitable allocations of the group portfolio, and finally demonstrate how these findings can be extended beyond distortion risk measures.

Keywords: Solvency II, Group Risk, Risk Sharing, Distortion Risk Measures, Value at Risk, Range Value at Risk.

1 Introduction

Capital requirements are a key instrument in the regulation of financial institutions. Their computation is typically based on two ingredients: stochastic balance sheet projections as a description of a firm's business, and monetary risk measures that capture the normative standards of a regulator. The question which risk measure to use for regulation is the topic of an ongoing discussion between academics and practitioners that began in the mid 1990s. Different properties of monetary risk measures have been suggested, and corresponding classes of risk measures have been identified and characterized.

Most of the scientific literature deals with convex risk measures: convex risk measures assign a lower risk measurement to a diversified position than to the non-diversified positions from which

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the diversified position is composed. If a convex risk measure is also positively homogeneous, it is subadditive; this property facilitates the delegation of risk limits from the company level to individual departments of the firm. Moreover, convex risk constraints are technically easier to handle in the context of portfolio optimization than non-convex constraints.

As this paper will show, risk measures that are not convex may have additional, even more serious deficiencies. The European regulatory framework for insurance firms, Solvency II, is built on the non-convex risk measure value at risk (V@R). In the current paper, we define a broader class of risk measures, called V@R-type risk measures, that includes V@R as a special case, and prove that a sophisticated firm can completely circumvent capital regulation that is based on these risk measures. The main idea is that a firm can adjust its structure over time and form a corporate group that consists of multiple legally separate entities. These entities are designed in such a way that they are regulated individually. Within a corporate group they are still owned and controlled by the same shareholders who seek to minimize the total capital required for the operation of the whole business.

Assuming that the future net asset value of the group is described by a random variable E, a sophisticated corporate group consisting of n subentities can split E into n parts. For this purpose, it needs to design suitable legally binding transfer agreements that produce an allocation $(E^1, E^2, \ldots, E^n)$ of the total net asset value among the subentities, satisfying $\sum_{i=1}^n E^i = E$. We show that for V@R-type risk measures and sufficiently many subentities the total capital requirement can thereby be reduced to the capital requirement of a corporate group with a deterministic future net asset value of esssup E, corresponding to the best case scenario. If the risk measure of one of the subentities is in addition strongly surplus sensitive – a property that we define in this paper – and if at the same time unlimited leverage is admissible, then the total capital requirement of the group can be reduced to levels that converge to minus infinity as the leverage of suitable subentities approaches infinity.

The paper is structured as follows: Section 2 reviews capital regulation and then recalls the family of distortion risk measures that includes value at risk (V@R), average value at risk (AV@R) and range value at risk (RV@R) as special cases. In Section 3 we first explain in detail the relationship between solvency capital minimization and optimal risk sharing. Second, we describe in the context of distortion risk measures how the total capital requirement of a corporate group can be reduced in order to circumvent capital regulation. Appropriate allocations for the corporate group are constructed. In the case of distortion risk measures, we provide explicit formulas in terms of mixtures of V@R for the total capital requirement of these allocations. Third, we generalize the main results beyond the case of distortion risk measures. All proofs are collected in the Appendix.

Literature

This paper is most closely related to Embrechts, Liu & Wang (2016) who investigate the risk sharing problem for a two-parameter class of quantile-based risk measures, called range value at risk (RV@R). This family, introduced by Cont, Deguest & Scandolo (2010), includes V@R and AV@R as limiting cases. Our paper, in contrast, provides a general picture on risk sharing for

V@R-type risk measures – a notion that is introduced in the current paper – and includes the main results of Embrechts et al. (2016) as special cases. A preliminary extension of the results of Embrechts et al. (2016) can also be found in Agirman (2016). For further related references on optimal risk sharing we refer to Embrechts et al. (2016). A description and analysis of corporate groups can be found in Keller (2007), Filipovic & Kupper (2008) and Haier, Molchanov & Schmutz (2015).

Some of our key arguments rely on the representation of distortion risk measures as mixtures of V@R, as described in Dhaene, Kukush, Linders & Tang (2012). For reviews on the theory of monetary risk measures, including distortion risk measures, we refer to Föllmer & Schied (2011) and Föllmer & Weber (2015). For further results on Choquet integrals and distortion risk measures see Choquet (1954), Dhaene, Vanduffel, Goovaerts, Kaas, Tang & Vyncke (2006), Denneberg (1994), Greco (1982), Schmeidler (1986), Song & Yan (2006), Song & Yan (2009*a*), Song & Yan (2009*b*), and Wang (1996). The current paper uses the same sign convention for risk measures as Embrechts et al. (2016).

2 Capital Regulation

2.1 Solvency II

A key instrument in the regulation of financial firms such as insurance companies and banks are solvency capital requirements. Their main role is to provide a buffer for potential losses that protects customers, policy holders and other counterparties. Solvency II is the regulatory framework that applies to European insurance companies. The computation of capital requirements is described in the Directive 2009/138/EC of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance – Solvency II (see European Commission (2009)):

> The Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases or, alternatively, that those undertakings will still be in a position, with a probability of at least 99.5 %, to meet their obligations to policy holders and beneficiaries over the following 12 months. That economic capital should be calculated on the basis of the true risk profile of those undertakings, taking account of the impact of possible risk-mitigation techniques, as well as diversification effects.

In a stylized manner, these principles can be formalized as follows: Consider an atomless probability space $(\Omega, \mathcal{F}, P)$ and a one period economy with dates $t = 0, 1$. Time 0 will be interpreted as today, time 1 as the one-year time horizon of Solvency II. Suppose that the solvency balance sheet of an insurance firm is available for $t = 0, 1$, e.g. computed from available data using an internal model. The value of the assets at time $t = 0, 1$ is denoted by A_t. We set L_t, $t = 0, 1$, for the value of the total liabilities to customers and other counterparties, net of economic capital. Economic capital is then computed as the difference of assets and liabilities, i.e. $E_t = A_t - L_t$, $t = 0, 1$. Observe that in this situation quantities at time 0 are deterministic while quantities at time 1 are random. For simplicity, we neglect the risk-less interest rates over this time horizon.[1]

Directive 2009/138/EC states that capital must be sufficient to prevent ruin with probability 99.5% on a one-year time horizon, i.e. $P(E_1 < 0) \le \alpha$ with $\alpha = 0.5\%$. Setting

$$SCR := V@R_\alpha(-\Delta E_1) \tag{1}$$

for $\Delta E_1 = E_1 - E_0$, we find conditions that are equivalent to the solvency requirement:

$$P(E_1 < 0) \le \alpha \quad \Leftrightarrow \quad -E_1 \in \mathcal{A}_{V@R_\alpha} \quad \Leftrightarrow \quad SCR \le E_0 \tag{2}$$

where $\mathcal{A}_{V@R_\alpha} = \{X \in L^\infty : P(X > 0) \le \alpha\}$ is the acceptance set of $V@R_\alpha$. Observe that – in contrast to Föllmer & Schied (2011), but consistent with Embrechts et al. (2016) – we make the convention that the argument of V@R counts losses positive and profits negative. We would like to point out that Directive 2009/138/EC provides an acceptance set for the company's capital at time $t = 1$. This is equivalent to verifying that the SCR is less than firm's capital at time $t = 0$ where the SCR is computed by the risk measure that corresponds to this acceptance set, evaluated at the random capital increment $-\Delta E_1$ (due to our sign convention).

While Solvency II limits the ruin probability at the one-year time horizon – corresponding to the acceptance set of V@R – this specific criterion can easily be replaced by others, i.e. by possibly more desirable acceptance sets. Then the modified SCR must be computed as the corresponding risk measure evaluated at the random capital increment $-\Delta E_1$. Examples include the Swiss Solvency test and Basel III. Both are based on AV@R, also called expected shortfall, conditional value at risk, tail value at risk, or tail conditional expectation. The next section recalls distortion risk measures that include both V@R and AV@R as special cases.

2.2 Distortion Risk Measures

To begin with, let $\mathcal{X}$ be the space of measurable and bounded functions on a measurable space $(\Omega, \mathcal{F})$.[2] A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is a monotone and cash-invariant function, see Föllmer & Schied (2011), Definition 4.1:

(i) *Monotonicity*: $\quad X, Y \in \mathcal{X}, X \le Y \;\Rightarrow\; \rho(X) \le \rho(Y)$

(ii) *Cash-Invariance*: $\quad X \in \mathcal{X}, m \in \mathbb{R} \;\Rightarrow\; \rho(X + m) = \rho(X) + m$

We use the convention that losses are counted as positive and gains as negative. A risk measure ρ is normalized, if $\rho(0) = 0$. It is distribution-based, if $\mathcal{X}$ is a space of random variables on some probability space $(\Omega, \mathcal{F}, P)$ and $\rho(X) = \rho(Y)$ whenever $P^X = P^Y$ for $X, Y \in \mathcal{X}$.

[1] For a discussion of this issue, we refer to Christiansen & Niemeyer (2014).

[2] The essential domain of risk measures, defined on larger spaces, relies on each risk measure itself. To keep the presentation simple, we first limit our attention to bounded measurable functions, but explain later – in Remark 13 – how larger domains may be chosen.

Any risk measure corresponds to its acceptance set, $\mathcal{A} = \{X \in \mathcal{X} : \rho(X) \leq 0\}$, from which it can be recovered as a capital requirement:

$$\rho(X) = \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}\}.$$

Using the notation of the previous section, solvency capital requirements are described as follows: If a regulator requires $-E_1 \in \mathcal{A}$, this is equivalent to $SCR \leq E_0$ with $SCR := \rho(-\Delta E_1)$.

We will now focus on a specific family of risk measures: distortion risk measures. Some results can be extended beyond this setting, see Section 3.3. Distortion risk measures form a subset of the family of comonotonic risk measures. The latter can be expressed as Choquet integrals with respect to capacities. Here we recall the main results that we will need.

Definition 1. *Two measurable functions X, Y on $(\Omega, \mathcal{F})$ are* comonotonic *if*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \quad \forall (\omega, \omega') \in \Omega \times \Omega.$$

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is comonotonic *if*

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

for comonotonic $X, Y \in \mathcal{X}$.

Remark 2. *(i) All comonotonic risk measures are positively homogeneous.*

(ii) V@R and AV@R are comonotonic. Cont et al. (2010) suggest an alternative to V@R and AV@R, called range value at risk (RV@R), which is a further example of a comonotonic risk measure. Letting $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$, they define

$$RV@R_{\alpha,\beta}(X) = \frac{1}{\beta} \int_{\alpha}^{\alpha+\beta} V@R_{\lambda}(X) d\lambda$$

for $X \in \mathcal{X}$. Like V@R, this is a non convex risk measure with an index of qualitative robustness[3] (IQR) of ∞, while AV@R is convex with IQR of 1. Observe that for convex risk measures, the IQR is at most 1. The limiting cases of $RV@R_{\alpha,\beta}$ correspond to $V@R_{\alpha}$ for $\beta \to 0$ and $AV@R_{\beta}$ for $\alpha \to 0$.

Definition 3. *(i) A mapping $c : \mathcal{F} \to [0, \infty)$ is called a* monotone set function *if it satisfies the following properties:*

(a) $c(\emptyset) = 0$.

(b) $A, B \in \mathcal{F}$, $A \subseteq B \Rightarrow c(A) \leq c(B)$.

If, in addition, $c(\Omega) = 1$, i.e. c is normalized, then c is called a capacity.

[3]For a precise definition see Krätschmer, Schied & Zähle (2014).

(ii) Let $X \in \mathcal{X}$. The Choquet integral *of X with respect to the monotone set function c is defined by*

$$\int X dc \;=\; \int_{-\infty}^{0} [c(X > x) - c(\Omega)]dx \;+\; \int_{0}^{\infty} c(X > x)dx$$

The Choquet integral coincides with the usual integral if c is a σ-additive probability measure.

Theorem 4. *A monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is comonotonic, if and only if there exists a capacity c on $(\Omega, \mathcal{F})$ such that*

$$\rho(X) = \int X dc.$$

Remark 5. *An important special case are distortion risk measures. In this case, the capacity is defined in terms of a distorted probability measure P. The resulting capacity is absolutely continuous with respect to P, but typically not additive.*

(i) An increasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$ is called a distortion function*. If P is a probability measure on $(\Omega, \mathcal{F})$, then*

$$c^g(A) := g(P[A]), \quad A \in \mathcal{F},$$

defines a capacity.

(ii) The corresponding distortion risk measure *$\rho^g(X) := \int X dc^g$ is coherent, if and only if g is concave.*

(iii) If an increasing function $g : [0,1] \to [0,\infty)$ with $g(0) = 0$ does not satisfy $g(1) = 1$, the equation $c^g(A) = g(P[A])$, $A \in \mathcal{F}$, still defines a monotone set function, but c^g is not normalized.

Definition 6. *Consider the class of distortion functions g such that*

$$\begin{aligned} g(x) = 0, &\quad \forall x \in [0, \alpha] \\ g(x) > 0, &\quad \forall x \in (\alpha, 1] \end{aligned}$$

for some $\alpha \in [0,1)$. The number α is called the parameter *of g, and*

$$\hat{g}(x) = \begin{cases} g(x+\alpha), & 0 \le x \le 1-\alpha \\ 1, & 1-\alpha < x \end{cases}$$

is the active part *of g. If the parameter $\alpha > 0$, then ρ^g is called a* V@R-type distortion risk measure*.*

V@R, AV@R and RV@R are distortion risk measures. V@R and RV@R are of V@R-type, AV@R is not. This is shown in the Table 1.

Risk Measure	$V@R_\alpha$	$AV@R_\beta$	$RV@R_{\alpha,\beta}$
g (x) =	$\begin{cases} 0, & 0 \leq x \leq \alpha \\ 1, & \alpha < x \end{cases}$	$\begin{cases} \frac{x}{\beta}, & 0 \leq x \leq \beta \\ 1, & \beta < x \end{cases}$	$\begin{cases} 0, & 0 \leq x \leq \alpha \\ \frac{x-\alpha}{\beta}, & \alpha < x \leq \alpha+\beta \\ 1, & \alpha+\beta < x \end{cases}$
Type	V@R-type	**Not** V@R-type	V@R-type

Table 1: Distortion functions for the risk measures V@R, AV@R and RV@R for $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$.

Remark 7. *Distortion risk measures can be expressed as mixtures of V@R. For arbitrary distortion functions the precise result is described in Dhaene et al. (2012). In this paper, we will focus only on the left-continuous case. Let* ρ^g *be defined as in Remark 5 for a left-continuous distortion function* g*, then*

$$\rho^g(X) \;=\; \int_{[0,1]} V@R_\lambda(X) g(d\lambda).$$

The integral on the right hand side of this equation is a Lebesgue-Stieltjes-integral with respect to the function g*.*

This representation provides an interpretation of the parameter α *of the distortion function* g *of a V@R-type distortion risk measure. The distortion risk measure evaluated at* $X \in \mathcal{X}$ *can be written as* $\rho^g(X) = \int_{[\alpha,1]} V@R_\lambda(X) g(d\lambda)$ *showing that this risk measurement does not depend on any properties of the tail of* X *beyond its V@R at level* α*.*

3 Group Risk Minimization

Financial institutions are typically owned by shareholders with limited liability. The free surplus that can be distributed as dividends to the shareholders is the economic capital less the SCR. Shareholders are thus interested in reducing the SCR via appropriate risk management techniques. Generalizing the results of Embrechts et al. (2016), we show that corporate group structures with sufficiently many entities allow a reduction of the total SCR of the group to the SCR of the best case scenario, if capital regulation is based on V@R-type distortion risk measures. We provide an upper bound for the optimal SCR for any number of entities in the corporate group and explicitly construct a group portfolio allocation that attains this bound. If the active parts of the considered distortion functions are concave, we show that the bound is sharp and the corresponding allocation is optimal. We also prove that, if losses and profits may be unbounded, the total capital requirement of the group may be reduced to any level, provided that one of the risk measures is strongly surplus sensitive. Finally, we demonstrate that our main results are not limited to the family of distortion risk measures. A reduction of the total SCR to the SCR of the best case scenario is in fact possible for all V@R-type risk measures in corporate groups that consist of sufficiently many entities; a reduction to an arbitrarily small level is admissible under conditions that we will specify.

3.1 The risk sharing problem of the group

Consider a financial corporation that consists of n entities that are all individually subject to capital regulation. The corporate group is, however, contractually structured in such a way that it serves the equity holders of a holding company that owns all other entities. Over short time horizons the number of entities n is fixed, but the corporation may adjust its structure over longer time horizons. Suppose that the total consolidated assets and liabilities at times $t = 0, 1$ are given by A_t and L_t, respectively. Total economic capital is, thus, given by $E_t = A_t - L_t$. We set $X = -E_1 = L_1 - A_1$.

The corporate group now uses at time $t = 0$ legally binding transfer agreements to modify the economic capital at time $t = 1$. In contrast to Filipovic & Kupper (2008), we do not assume that these transfers are constructed as linear portfolios of a finite family of standardized capital transfer products. Instead we suppose that transfer agreements are contingent claims that admit any reallocation of total capital among the n entities of the group. The resulting allocation will be denoted by $(E_1^i)_{i=1,2,\dots,n}$. We set $X^i = -E_1^i$, $i = 1, 2, \dots, n$, and observe that

$$X = \sum_{i=1}^{n} X^i.$$

We suppose that the solvency capital requirement SCR^i of entity $i = 1, 2, \dots, n$ is computed on the basis of a risk measure ρ^i, i.e.

$$SCR^i = E_0^i + \rho^i(X^i),$$

where E_0^i refers to the economic capital of entity i at time 0. It holds that $\sum_{i=1}^n E_0^i = E_0$. The total solvency capital requirement of the group is thus given by

$$\sum_{i=1}^{n} SCR^i = E_0 + \sum_{i=1}^{n} \rho^i(X^i).$$

For a fixed number n of entities the problem of the corporate group consists in the design of optimal transfers that minimize $\sum_{i=1}^n \rho^i(X^i)$. We will, in particular, show that for V@R-type risk measures and sufficiently large n, the corporate group can find a capital allocation such that

$$\sum_{i=1}^{n} \rho^i(X^i) = \operatorname{essinf} X = -\operatorname{esssup} E_1,$$

corresponding to the best case scenario. If one of the risk measures is surplus sensitive (a property that we will define later) and if entities may hold arbitrarily large liabilities, then the total risk can even be made arbitrarily small.

3.2 Risk sharing for V@R-type distortion risk measures

We now consider the optimal risk sharing problem

$$\square_{i=1}^{n}\rho^{i}\,(X) \;:=\; \inf\left\{\sum_{i=1}^{n}\rho^{i}(X^{i}):\;\sum_{i=1}^{n}X^{i}=X,\quad X^{1},X^{2},\dots,X^{n}\in L^{\infty}\right\}. \tag{3}$$

The following theorem provides an upper bound to the solution and an allocation that attains this bound.

Theorem 8. *Let $X\in L^{\infty}$ and $n\in\mathbb{N}$. By $g^{1},g^{2},\dots,g^{n}$ we denote left-continuous distortion functions with finitely many jumps and parameters $\alpha_{1},\alpha_{2},\dots,\alpha_{n}\in[0,1)$ and define $d=\sum_{i=1}^{n}\alpha_{i}$. We set $\rho^{i}=\rho^{g_{i}}$, i.e. ρ^{i} is the distortion risk measure associated with the distortion function g^{i}, $i=1,2,\dots,n$. Define the left-continuous functions*

$$f=\min\{\widehat{g^{1}},\widehat{g^{2}},\dots,\widehat{g^{n}}\},\qquad g(x)=\begin{cases}0, & 0\le x\le d\wedge 1,\\ f(x-d), & d\wedge 1<x\le 1\end{cases}$$

Note that $g\equiv 0$, if $d\geq 1$. We set $V@R_{\lambda}:=V@R_{1}=\operatorname{essinf}$ for $\lambda\geq 1$.

(i) There exists a collection of disjoint intervals $(a_{k}^{u},a_{k}^{o})\subseteq[0,1]$, $k\in\mathbb{N}$, with $\bigcup_{k\in\mathbb{N}}[a_{k}^{u},a_{k}^{o}]=[0,1]$ and a mapping $j:\mathbb{N}\to\{1,2,\dots,n\}$ such that for all $k\in\mathbb{N}$ and $x\in(a_{k}^{u},a_{k}^{o}]$: $f(x)=\widehat{g^{j(k)}}(x)$.

(ii) There exist $X^{1},X^{2},\dots,X^{n}\in L^{\infty}$ such that $\sum_{i=1}^{n}X^{i}=X$ and

$$\sum_{i=1}^{n}\rho^{i}(X^{i})=\int_{[0,1]}V@R_{\lambda}(X-\operatorname{essinf}\,X)g(d\lambda)+\operatorname{essinf}\,X.$$

If $d\geq 1$, this equation can be simplified and we obtain

$$\sum_{i=1}^{n}\rho^{i}(X^{i})=\operatorname{essinf}\,X.$$

(iii) The allocation $(X^{i})_{i=1,2,\dots,n}$ can be constructed as follows. Let

$$Y\;:=\;X-\operatorname{essinf}\,X\;\geq\;0.$$

For $k\in\mathbb{N}$ we define the following random variables:

$$\begin{aligned}\tilde{X}^{k}\;&=\;\Big(V@R_{d+a_{k}^{u}}(Y)-V@R_{d+a_{k}^{o}}(Y)\Big)\cdot\mathbf{1}_{\{V@R_{d}(Y)\geq Y>V@R_{d+a_{k}^{u}}(Y)\}}\\ &\quad+\Big(Y-V@R_{d+a_{k}^{o}}(Y)\Big)\cdot\mathbf{1}_{\{V@R_{d+a_{k}^{u}}(Y)\geq Y>V@R_{d+a_{k}^{o}}(Y)\}}\end{aligned}$$

For $i = 1, 2, \ldots, n$ we set

$$X^i \quad = \quad \sum_{k: i = j(k)} \tilde{X}^k + Y \cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{i-1} \alpha_l}(Y) \geq Y > V@R_{\sum_{l=1}^{i} \alpha_l}(Y)\}} + \frac{\text{essinf } X}{n} \tag{4}$$

If $d \geq 1$, this equation can be simplified and we obtain

$$X^i \quad = \quad Y \cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{i-1} \alpha_l}(Y) \geq Y > V@R_{\sum_{l=1}^{i} \alpha_l}(Y)\}} + \frac{\text{essinf } X}{n} \tag{5}$$

Proof. See Section A.1. □

Corollary 9. *Suppose that the conditions of Theorem 8 hold. The solution to the optimal risk sharing problem* (3) *is bounded by*

$$\square_{i=1}^n \rho^i\,(X) \leq \int_{[0,1]} V@R_\lambda(X - \text{ essinf } X) g(d\lambda) + \text{essinf } X.$$

In particular, if $d \geq 1$, this bound is equal to the total risk of the best case scenario essinf X *of* X *evaluated by an arbitrary normalized risk measure, i.e.*

$$\square_{i=1}^n \rho^i\,(X) \leq \text{ essinf } X.$$

Proof. See Section A.2. □

Let us now specify additional assumptions such that the upper bound of Corollary 9 is at the same time a lower bound and thus equal to the value of the optimal risk sharing problem.

Theorem 10. *Suppose that the conditions of Theorem 8 hold. In addition, assume that $d < 1$ and $g^i(1 - d + \alpha_i) = 1$ for $i = 1, 2, \ldots, n$, and that the active parts of the distortion functions $g^1, g^2, \ldots, g^n$ are concave. Then the allocation defined in eq.* (4) *provides a solution to the optimal risk sharing problem* (3) *and*

$$\square_{i=1}^n \rho^i\,(X) = \int_{[0,1]} V@R_\lambda(X) g(d\lambda).$$

Proof. See Section A.3. □

Theorem 8, Corollary 9 and Theorem 10 provide an important perspective on capital regulation based on V@R-type distortion risk measures. They show that (if risk is measured by a normalized risk measure and the group consists of sufficiently many entities) the total capital requirement can be made equal to the capital requirement of the best case scenario of the group, i.e. essinf $X = -\text{esssup } E_1$. This quantity is an upper bound to the solution of the optimal risk sharing problem. Downside risk can thus completely be hidden within corporate group structures. V@R is a special case of a V@R-type distortion risk measure, and our observations apply to Solvency II. In contrast, they do not apply to the Swiss Solvency Test that uses the coherent risk measure AV@R as the basis for capital regulation.

Observe that the allocation (X^i) in equation (5) is bounded from below by $\frac{\operatorname{essinf} X}{n}$ and from above by esssup $Y + \frac{\operatorname{essinf} X}{n}$. We could thus restrict the admissible allocations to those that are bounded by suitable fixed constants and still obtain the results stated above. If no bounds are imposed, the situation can even be more serious from the point of view of capital regulation. We will show in Example 11, Theorem 12, and Remark 13 that the total capital requirement of the group can be further reduced, if no bounds are imposed on admissible profits and losses of group entities. In these cases, risk sharing can be used to make the total risk $\sum_{i=1}^n \rho^i(X^i)$ arbitrarily small for appropriately chosen allocations, i.e. smaller than $-m$ for any $m \in \mathbb{N}$. In Example 11 and Theorem 12 very large losses of one entity may occur due to transfer agreements with an other entity that experiences large profits in the corresponding scenarios. Remark 13, in contrast, parallels the results of Theorem 8 and Corollary 9, but in the case where X itself is unbounded.

Example 11. *Let $(\Omega, \mathcal{F}, P)$ be a probability space without atoms. Consider a corporate group of $n = 2$ entities with risk measures $\rho^1 = \rho^2 = RV@R_{\frac{1}{4},\frac{3}{4}}$. We will show that $\square_{i=1}^2 \rho^i\ (0) = -\infty$. To this end, let $A_1, A_2 \subseteq \Omega$ be a partition of Ω such that $P(A_1) = \frac{1}{8}$, $P(A_2) = \frac{7}{8}$. Let $m \in \mathbb{N}$ be arbitrary, and set $X^1 := 6m \cdot \mathbf{1}_{A_1}$, $X^2 := -6m \cdot \mathbf{1}_{A_1}$. Then $X^1 + X^2 = 0$, $\rho^1(X^1) = 0$, $\rho^2(X^2) = -6m \cdot \frac{1}{8} \cdot \frac{4}{3} = -m$. Thus, $\square_{i=1}^2 \rho^i\ (0) \leq -m$ for any $m \in \mathbb{N}$.*

We now provide a theorem that characterizes the situation of the previous example on a general level.

Theorem 12. *Suppose that the conditions of Theorem 8 hold and assume that there exists $i \in \{1, 2, \ldots, n\}$ such that $g^i(1 - d + \alpha_i) < 1$. Then*

$$\square_{i=1}^n \rho^i\ (X) \ = \ -\infty.$$

Proof. See Section A.4. $\square$

From a regulatory point of view, under the conditions of the last theorem, capital regulation can completely be circumvented in corporate groups: the total downside risk measurements are not bounded from below anymore. One should, however, note that group allocations with arbitrarily small total risk are associated with arbitrarily large losses and profits of some of the entities of the group. For insurance groups, the implementation of the required transfer agreements might not be realistic, if the admissible leverage is bounded for each entity. This means that in practice Theorem 12 is less relevant for capital regulation than Theorem 8, Corollary 9 and Theorem 10. It stresses, however, potential problems that might occur if significant leverage and V@R-type distortion risk measures are used together.

Remark 13. *We have been considering the risk sharing problem for bounded $X \in L^\infty$, but this restriction is not necessary. Suppose now that X is an arbitrary random variable. If X is bounded from below, i.e.* essinf $X > -\infty$, *it is not difficult to verify that the results of Theorem*

8 are still valid. This is due to the fact that in Theorem 8 total losses X beyond $V@R_d(X)$ are allocated to the positions $(X^i)_{i=1,2,\dots,n}$ such that they do not influence the risk measurements $(\rho^i(X^i))_{i=1,2,\dots,n}$.

Next, let us assume that X is not bounded from below anymore, i.e. $\operatorname{essinf} X = -\infty$. *Consider the case $d \geq 1$ and n large enough. Then* $\operatorname{essinf}\,(X \vee (-k)) = -k$ *for $k \geq 0$. By the monotonicity of risk measures,*

$$\square_{i=1}^n \rho^i(X) \;\leq\; \square_{i=1}^n \rho^i(X \vee (-k)) \;=\; -k \;\stackrel{k\to\infty}{\longrightarrow}\; -\infty.$$

We thus obtain a result that is analogous to the situation of Theorem 12: the total downside risk measurement is not bounded from below, if the best case is unbounded.

3.3 Risk sharing for V@R-type risk measures

So far, we have been focussing on distortion risk measures. For $d < 1$ we do, indeed, need this particular structure to compute the exact total risk of the allocation defined in eq. (4). This allocation provides firstly an upper bound for the total risk of the optimal risk sharing problem and secondly a solution in the case of concave active parts. In the case of $d \geq 1$ the allocation defined in eq. (5) provides a bound, and it turns out that this result is not limited to the family of distortion risk measures. The next theorem provides a precise statement. In addition, we will also be able to generalize Theorem 12 beyond the case of distortion risk measures.

Definition 14. *A distribution-based risk measure $\rho : L^\infty \to \mathbb{R}$ is a* V@R-type risk measure *with parameter $\alpha > 0$, if*

$$\rho(X) \;=\; \rho\left(X \cdot \mathbf{1}_{\{V@R_\alpha(X) \geq X\}} + V@R_\alpha(X) \cdot \mathbf{1}_{\{V@R_\alpha(X) < X\}}\right) \qquad (X \in L^\infty).$$

Remark 15. *Obviously, any V@R-type distortion risk measure with parameter $\alpha > 0$ is a V@R-type risk measure with the same parameter. This follows immediately from Remark 7.*

Theorem 16. *Let $X \in L^\infty$, $\rho^1, \rho^2, \dots, \rho^n$ be V@R-type risk measures with parameters $\alpha_1, \alpha_2, \dots, \alpha_n$, and the allocation $(X^i)_{i=1,2,\dots,n}$ be given according to equation (5). If $d = \sum_{i=1}^n \alpha_i \geq 1$, then*

$$\square_{i=1}^n \rho^i\,(X) \leq \sum_{i=1}^n \rho^i(X^i) \;=\; \sum_{i=1}^n \rho^i(0) \;+\; \operatorname{essinf} X.$$

In particular, if the risk measures ρ^i are normalized, i.e. $\rho^i(0) = 0$, $i = 1, 2, \dots, n$, then the minimal total risk is bounded by the risk of the best case scenario $\operatorname{essinf} X$ *of X evaluated by an arbitrary normalized risk measure.*

Proof. See Section A.5. □

Remark 17. *Arguments analogous to those in Remark 13 show that the result of Theorem 16 is not limited to positions in the space L^∞, but holds for larger spaces of random variables. If* $\operatorname{essinf} X = -\infty$, *the total risk measurement can be made arbitrarily small for suitable risk sharing allocations.*

We finally show how Theorem 12 may be generalized beyond the case of distortion risk measures.

Definition 18. *A distribution-based risk measure $\rho : L^\infty \to \mathbb{R}$ is* surplus sensitive at level $\alpha > 0$, *if*

$$\rho(X) > \rho(X - m \cdot \mathbf{1}_{\{V@R_{1-\alpha}(X) \geq X\}} =: h_x(m)$$

for any $m > 0$.

If, in addition, $h_X(m) \to -\infty$ as $m \to \infty$, then ρ is strongly surplus sensitive at level $\alpha > 0$.

Remark 19. *(i) A risk measure that is surplus sensitive is not necessarily strongly surplus sensitive at the same level. An example is the entropic risk measure. We consider the special case $\rho(X) = \log E(e^X)$. Setting $\alpha = 1/2$, we compute h_X for a random variable X with $P(X = \log 2) = 1/2$ and $P(X = 0) = 1/2$. Then $h_X(m) = \log E(e^{X - m \cdot \mathbf{1}_{V@R_{1/2}(X) \geq X}}) = \log(\frac{1}{2}e^{\log 2} + \frac{1}{2}e^{-m}) = \log(1 + \frac{e^{-m}}{2}) \overset{m\to\infty}{\longrightarrow} 0 > -\infty$.*

(ii) In contrast, any distortion risk measure ρ^g with distortion function g such that $g(x) < 1$ for $x < 1$ is strongly surplus sensitive at any level $d > 0$. Another example are expectiles with acceptance set $\left\{X \in L^\infty : \quad \frac{E(X^-)}{E(X^+)} \geq \gamma\right\}$ for $\gamma > 0$.

Theorem 20. *Let $X \in L^\infty$ and $\rho^1, \rho^2, \ldots, \rho^n$ be V@R-type risk measures with parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$. Set $d = \sum_{i=1}^n \alpha_i$. If ρ^{n+1} is strongly surplus sensitive at level d, then*

$$\square_{i=1}^{n+1} \rho^i(X) = -\infty.$$

Proof. See Section A.6. $\square$

Remark 21. *Theorem 12 can be seen as a corollary to Theorem 20. But the direct proof of Theorem 12 in Section A.4 computes in addition explicitly the exact total risk of the allocations that were considered to bound the value of the risk sharing problem* (3) *from above.*

In contrast to Theorem 16, the practical relevance of Theorem 20 might be limited by the fact that the required allocations are associated with arbitrarily large losses and profits, i.e. leverage is unbounded. Nevertheless, like Theorem 12, it highlights the problems that arise when V@R-type risk measures and leverage are combined.

4 Conclusion

Non-coherent risk measures have frequently been criticized during the last twenty years. The present paper presents another challenge to capital regulation with non-coherent risk measures of V@R-type: sophisticated firms can hide their downside risk within corporate groups by designing suitable intra-group transfers. In this paper, these group transfers are specified as derivatives on the stochastic balance sheet of the group. Future research needs to express these as contingent claims on tradeable securities (or, at least, on quantities that cannot easily be manipulated by the managers of the firm).

The feasibility of the discovered capital reduction strategy relies on the fact that capital requirements are charged separately for each entity of the group. Intra-group transfers could not reduce the total capital requirement, if the latter was defined on the basis of a consolidated balance sheet. Governments could make this mandatory. Such a requirement would need a legal framework that defines a corporate group in terms of joint interests of the owners of different entities and instruments that allow coordinated governance. At the same time, collusion between entities in separate corporate groups would have to be forbidden and prevented. However, even if a consolidated approach to group risk management was required, serious problems would occur. Groups consist of legally separate firms with limited liability. If capital requirements are computed on the basis of a consolidated balance sheet, groups could exploit multiple limited liability options associated with their subentities via appropriate strategic defaults at the expense of third parties, optimizing their own gains with prior intra-group transfers. Consolidated balance sheets do not reflect these possibilities. Our argument indicates that consolidated balance sheets should be accompanied by joint liability of the group and *not* be combined with limited liability of subentities.

In summary, from a regulatory point of view V@R-type risk measures and corporate groups of legally separate entities are not compatible. If one recalls other well-known deficiencies of non-coherent risk measures, V@R-type risk measures do not seem to be an optimal ingredient to the regulation of insurance companies.

A Proofs

A.1 Proof of Theorem 8

Proof. (i): Assume first that $g^1, g^2, \dots, g^n$ are continuous. We define recursively the following closed sets:

$$\begin{aligned} A_1 &= \{\widehat{g^1} = f\} \\ A_i &= \{\widehat{g^i} = f\} \setminus (A_1^o \cup \dots \cup A_{i-1}^o), \quad i = 2, 3, \dots, n. \end{aligned}$$

Since $\bigcup_{i=1}^n A_i = [0,1]$, Baire's category theorem (see e.g. Thm. 3.46 in Aliprantis & Border (2006)) implies that $A = \bigcup_{i=1}^n A_i^o$ is dense in $[0,1]$. By construction, $A_i^o \cap A_j^o = \emptyset$ for all $i \neq j$. Any open set in $[0,1]$ can be represented as the union of a countable collection of disjoint open intervals. Let (a_k^u, a_k^o), $k \in \mathbb{N}$, be such a collection of intervals that represents the set A. Since the sets $(A_i)_{i=1,2,\dots,n}$ are disjoint, for each $k \in \mathbb{N}$ there exists a unique $i \in \{1, 2, \dots, n\}$ such that $(a_k^u, a_k^o) \subseteq A_i$. We define $j(k) = i$ in this case and note that for all $x \in (a_k^u, a_k^o)$ we have $f(x) = \widehat{g^i}(x)$ by construction of the set A_i. Since A is dense in $[0,1]$, it follows that $\bigcup_{k \in \mathbb{N}} [a_k^u, a_k^o] = [0,1]$.

Second, if the distortion functions have finitely many jumps, the location of these jumps defines a partition of $[0,1]$ into finitely many subintervals. Then the argument above can separately be applied to the closure of each of these subintervals and the continuous extensions of the restrictions of the considered functions to the interior of these subintervals. This implies the existence of a collection of disjoint intervals $(a_k^u, a_k^o) \subseteq [0,1]$ with $\bigcup_{k \in \mathbb{N}} [a_k^u, a_k^o] = [0,1]$ and a mapping $j : \mathbb{N} \to \{1, 2, \dots, n\}$ such that for all $k \in \mathbb{N}$ and $x \in (a_k^u, a_k^o)$: $f(x) = \widehat{g^{j(k)}}(x)$.

Finally, we need to verify the claimed equality at right endpoints at the intervals, i.e. $f(a_k^o) = \widehat{g^{j(k)}}(a_k^o)$. This follows from the left-continuity of the functions $\widehat{g^1}, \widehat{g^2}, \dots, \widehat{g^n}$ and f.
(ii) & (iii): We first show that $X = \sum_{i=1}^n X^i$. Note that equality is always understood modulo P-nullsets. To this end, observe that $\tilde{X}^k \geq 0$, $k \in \mathbb{N}$, thus $\sum_{k \in \mathbb{N}} \tilde{X}^k$ is well-defined. We have

$$\{V@R_d(Y) \geq Y\} = \bigcup_{k \in \mathbb{N}} \left\{ V@R_{d + a_k^u}(Y) \geq Y > V@R_{d + a_k^o}(Y) \right\}.$$

For $m \in \mathbb{N}$ we observe that $\min\{a_k^u > a_m^u : \ k \in \mathbb{N}\} = a_m^o$ and for $\omega \in \{V@R_{d + a_m^u}(Y) \geq Y > V@R_{d + a_m^o}(Y)\}$ we compute

$$\begin{aligned} \sum_{k \in \mathbb{N}} \tilde{X}^k(\omega) &= \sum_{k \in \mathbb{N},\, a_k^u > a_m^u} \Big(V@R_{d + a_k^u}(Y) - V@R_{d + a_k^o}(Y) \Big) + Y(\omega) - V@R_{d + a_m^o}(Y) \\ &= V@R_{d + a_m^o}(Y) - \underbrace{V@R_1(Y)}_{=0} + Y(\omega) - V@R_{d + a_m^o}(Y) = Y(\omega). \end{aligned}$$

For $\omega \in \{V@R_d(Y) < Y\}$, we get $\sum_{k \in \mathbb{N}} \tilde{X}^k(\omega) = 0$.

Thus,

$$\sum_{k\in\mathbb{N}} \tilde{X}^k = Y \cdot \mathbf{1}_{\{V@R_d(Y)\geq Y\}}. \tag{6}$$

This implies that

$$\begin{aligned}\sum_{i=1}^n X^i &= \sum_{k\in\mathbb{N}} \tilde{X}^k + Y \cdot \sum_{i=1}^n \mathbf{1}_{\{V@R_{\sum_{l=1}^{i-1}\alpha_l}(Y) \geq Y > V@R_{\sum_{l=1}^{i}\alpha_l}(Y)\}} + \text{essinf } X \\ &= Y \cdot \mathbf{1}_{\{V@R_d(Y)\geq Y\}} + Y \cdot \mathbf{1}_{\{V@R_d(Y)<Y\}} + \text{essinf } X = Y + \text{essinf } X = X.\end{aligned}$$

Next, we analyze the mapping $\lambda \mapsto V@R_\lambda(X^i)$. First, observe that by construction $\tilde{X}^k \geq 0$ for $k \in \mathbb{N}$, thus

$$\|\sum_{k:i=j(k)} \tilde{X}^k\|_\infty \leq \|\sum_{k\in\mathbb{N}} \tilde{X}^k\|_\infty \overset{eq.(6)}{=} \|Y \cdot \mathbf{1}_{\{V@R_d(Y)\geq Y\}}\|_\infty \leq V@R_d(Y).$$

This implies for $0 \leq \lambda < \alpha_i$ that

$$V@R_\lambda(X^i) = V@R_{\lambda+\sum_{l=1}^{i-1}\alpha_l}(Y) + \frac{\text{essinf } X}{n}.$$

For $\lambda \geq \alpha_i$, we get

$$V@R_\lambda(X^i) = V@R_{\lambda-\alpha_i}\left(\sum_{k:i=j(k)} \tilde{X}^k\right) + \frac{\text{essinf } X}{n}. \tag{7}$$

Observe that $\tilde{X}^k$, $k \in \mathbb{N}$, are decreasing functions of $Y \cdot \mathbf{1}_{\{V@R_d(Y)\geq Y\}}$, thus comonotone. Since the value at risk is comonotone additive, we obtain for any $K \in \mathbb{N}$ that

$$V@R_\lambda\left(\sum_{k\leq K:i=j(k)} \tilde{X}^k\right) = \sum_{k\leq K:i=j(k)} V@R_\lambda(\tilde{X}^k), \quad \lambda \geq 0.$$

By construction, $\tilde{X}^k \geq 0$ for any $k \in \mathbb{N}$. This implies that $(\sum_{k\leq K:i=j(k)} \tilde{X}^k)_K$ is an increasing sequence bounded by $V@R_d(Y)$ that converges almost surely and thus in distribution to $\sum_{k:i=j(k)} \tilde{X}^k$. Since $V@R_\lambda$ is weakly continuous, the left hand side of the equation converges to $V@R_\lambda\left(\sum_{k:i=j(k)} \tilde{X}^k\right)$ as $K \to \infty$. The summands on the right hand side are all positive which implies convergence for $K \to \infty$. We thus obtain

$$V@R_\lambda\left(\sum_{k:i=j(k)} \tilde{X}^k\right) = \sum_{k:i=j(k)} V@R_\lambda(\tilde{X}^k), \quad \lambda \geq 0. \tag{8}$$

Analogous arguments show that

$$V@R_\lambda\left(\sum_{k\in\mathbb{N}}\tilde{X}^k\right)=\sum_{k\in\mathbb{N}}V@R_\lambda(\tilde{X}^k),\quad \lambda\geq 0. \tag{9}$$

The value at risk of $\tilde{X}^k$, $k\in\mathbb{N}$, is given by

$$V@R_\lambda(\tilde{X}^k)=\begin{cases} V@R_{d+a_k^u}(Y)-V@R_{d+a_k^o}(Y), & \lambda<a_k^u\\ V@R_{d+\lambda}(Y)-V@R_{d+a_k^o}(Y), & a_k^u\leq\lambda<a_k^o\\ 0, & a_k^o<\lambda\end{cases}$$

This implies by Theorem 6 in Dhaene et al. (2012) that

$$\begin{aligned}\int_0^1 V@R_\lambda(\tilde{X}^k)d\widehat{g^{j(k)}}(\lambda) &= \widehat{g^{j(k)}}(a_k^u+)\cdot\Big(V@R_{d+a_k^u}(Y)-V@R_{d+a_k^o}(Y)\Big)+\int_{(a_k^u,a_k^o]}V@R_\lambda(\tilde{X}^k)d\widehat{g^{j(k)}}(\lambda)\\ &= f(a_k^u+)\cdot\Big(V@R_{d+a_k^u}(Y)-V@R_{d+a_k^o}(Y)\Big)+\int_{(a_k^u,a_k^o]}V@R_\lambda(\tilde{X}^k)df(\lambda)\\ &= \int_0^1 V@R_\lambda(\tilde{X}^k)df(\lambda). \end{aligned}\tag{10}$$

Finally, we obtain

$$\begin{aligned}\sum_{i=1}^n\rho^i(X^i) &= \sum_{i=1}^n\int_0^1 V@R_\lambda(X^i)dg^i(\lambda)=\sum_{i=1}^n\int_{[\alpha_i,1]}V@R_\lambda(X^i)dg^i(\lambda)\\ &\overset{eq.\ (7)\&(8)}{=} \text{essinf } X+\sum_{i=1}^n\int_{[\alpha_i,1]}\sum_{k:i=j(k)}V@R_{\lambda-\alpha_i}(\tilde{X}^k)dg^i(\lambda)\\ &= \text{essinf } X+\sum_{k\in\mathbb{N}}\int_{[\alpha_{j(k)},1]}V@R_{\lambda-\alpha_{j(k)}}(\tilde{X}^k)dg^{j(k)}(\lambda)\quad\text{(by Monotone Convergence Thm.)}\\ &= \text{essinf } X+\sum_{k\in\mathbb{N}}\int_{[0,1-\alpha_{j(k)}]}V@R_\lambda(\tilde{X}^k)d\widehat{g^{j(k)}}(\lambda)\\ &= \text{essinf } X+\sum_{k\in\mathbb{N}}\int_0^1 V@R_\lambda(\tilde{X}^k)d\widehat{g^{j(k)}}(\lambda)\qquad\text{(since } \widehat{g^{j(k)}}(x)=1 \text{ for } x\geq 1-\alpha_{j(k)})\\ &\overset{eq.\ (10)}{=} \text{essinf } X+\sum_{k\in\mathbb{N}}\int_0^1 V@R_\lambda(\tilde{X}^k)df(\lambda)\\ &= \text{essinf } X+\int_0^1\sum_{k\in\mathbb{N}}V@R_\lambda(\tilde{X}^k)df(\lambda)\qquad\text{(by Monotone Convergence Thm.)}\end{aligned}$$

$$
\begin{aligned}
&\overset{(9)}{=} && \operatorname{essinf} X+\int_0^1 V@R_\lambda(\sum_{k\in\mathbb{N}}\tilde{X}^k)df(\lambda)\\
&\overset{eq.(6)}{=} && \operatorname{essinf} X+\int_0^1 V@R_\lambda\left(Y\cdot \mathbf{1}_{\{V@R_d(Y)\geq Y\}}\right)df(\lambda)\\
&= && \operatorname{essinf} X+\int_0^1 V@R_\lambda\left(Y\right)dg(\lambda)\\
&= && \operatorname{essinf} X+\int_0^1 V@R_\lambda\left(X-\operatorname{essinf} X\right)dg(\lambda)
\end{aligned}
$$

□

A.2 Proof of Corollary 9

Proof. Obviously,

$$\square_{i=1}^n \rho^i\ (X)\ \leq\ \sum_{i=1}^n \rho^i(X^i)$$

for the allocation defined in equation (4). Thus, both claims follow from Theorem 8. □

A.3 Proof of Theorem 10

Proof. Note first that $1=g^i(1-d+\alpha_i)=\widehat{g^i}(1-d)$ for $i=1,2,\ldots,n$, thus $g(1)=f(1-d)=\min\{\widehat{g^1}(1-d),\widehat{g^2}(1-d),\ldots,\widehat{g^n}(1-d)\}=1$.

Observe that the inequality "$\leq$" follows from Corollary 9, since $g(1)=1$ and V@R is cash-invariant. We present a proof for the inequality "$\geq$".[4] We prove the statement by induction over the number n of distortion functions $g^1,g^2,\ldots,g^n$.

Consider first two distortion functions g^i, $i=1,2$, i.e. $n=2$. In this case, $d=\alpha_1+\alpha_2$. Given any $X^1,X^2\in L^\infty$ with $X^1+X^2=X$ we construct $Y^1,Y^2\in L^\infty$ such that

$$
\begin{aligned}
\rho^1(X^1)+\rho^2(X^2) &\overset{(11)}{=} \rho^{\widehat{g^1}}(Y^1)+\rho^{\widehat{g^2}}(Y^2) \overset{(12)}{\geq} \rho^f(Y^1)+\rho^f(Y^2)\\
&\overset{(13)}{\geq} \rho^f(Y^1+Y^2) \overset{(14)}{\geq} \int_{[0,1]} V@R_\lambda(X^1+X^2)g(d\lambda).
\end{aligned}
$$

Observe that by Theorem 6 in Dhaene et al. (2012) we have that $\int V@R_\lambda(X^1+X^2)g(d\lambda)=\rho^g(X^1+X^2)$, since $g(1)=1$.
We will first specify Y^1,Y^2 and then verify the inequalities (11) – (14).

To begin with, observe that $\tilde{X}^1=X^1-\operatorname{essinf} X^1\geq 0$, $\tilde{X}^2=X^2-\operatorname{essinf} X^2\geq 0$. If $\rho^1(\tilde{X}^1)+\rho^2(\tilde{X}^2)\geq\rho^g(\tilde{X}^1+\tilde{X}^2)$, we add $\operatorname{essinf} X^1+\operatorname{essinf} X^2$ to both sides in order to obtain

[4] A preliminary draft version of this proof can be found in the B.Sc. thesis Agirman (2016) that was supervised by the author of this paper.

by cash-invariance that $\rho^1(X^1) + \rho^2(X^2) \geq \rho^g(X^1 + X^2)$. We may thus assume w.l.og. that $X^1, X^2 \geq 0$ and will do so henceforth.

Set $Y^1 = X^1 \cdot \mathbf{1}_{\{V@R_{\alpha_1}(X^1) \geq X^1\}}$, $Y^2 = X^2 \cdot \mathbf{1}_{\{V@R_{\alpha_2}(X^2) \geq X^2\}}$. Then

$$V@R_\lambda(Y^i) = \begin{cases} V@R_{\alpha_i+\lambda}(X^i), & \lambda < 1 - \alpha_i, \\ 0, & 1 - \alpha_i \leq \lambda. \end{cases}$$

This implies by Theorem 6 in Dhaene et al. (2012) that

$$\begin{aligned} & \rho^1(X^1) + \rho^2(X^2) = \int V@R_\lambda(X^1) dg_1(\lambda) + \int V@R_\lambda(X^2) dg_2(\lambda) \\ = & \int V@R_{\alpha_1+\lambda}(X^1) d\hat{g}_1(\lambda) + \int V@R_{\alpha_2+\lambda}(X^2) d\hat{g}_2(\lambda) = \int V@R_\lambda(Y^1) d\hat{g}_1(\lambda) + \int V@R_\lambda(Y^2) d\hat{g}_2(\lambda) \\ = & \rho^{\widehat{g^1}}(Y^1) + \rho^{\widehat{g^2}}(Y^2), \text{ i.e. equation (11).} \end{aligned}$$

Now observe that by definition $\widehat{g^i} \geq f$, $i = 1, 2$. Thus, $\rho^{\widehat{g^1}}(Y^1) + \rho^{\widehat{g^2}}(Y^2) \geq \rho^f(Y^1) + \rho^f(Y^2)$, i.e. inequality (12). Since f is concave, it follows moreover that $\rho^f(Y^1) + \rho^f(Y^2) \geq \rho^f(Y^1 + Y^2)$, i.e. inequality (13).

Let $A_i := \{V@R_{\alpha_i}(X^i) < X^i\}$, $i = 1, 2$. Then $P(A_i) \leq \alpha_i$, $i = 1, 2$. Observe that $Y^i = X^i$ on the complement A_i^c of A_i, $i = 1, 2$. For $x \in \mathbb{R}$ we get

$$\begin{aligned} & P\{Y^1 + Y^2 > x\} \geq P(\{Y^1 + Y^2 > x\} \cap (A_1 \cup A_2)^c) \\ = \; & P(\{X^1 + X^2 > x\} \cap (A_1 \cup A_2)^c) \geq P\{X^1 + X^2 > x\} - P(A_1 \cup A_2) \\ \geq \; & P\{X^1 + X^2 > x\} - (\alpha_1 + \alpha_2) = P\{X^1 + X^2 > x\} - d. \end{aligned}$$

Since $P\{Y^1 + Y^2 > x\} \geq 0$, we get

$$P\{Y^1 + Y^2 > x\} \geq (P\{X^1 + X^2 > x\} - d) \vee 0 = (P\{X^1 + X^2 > x\} \vee d) - d.$$

We have $Y^1 + Y^2 \geq 0$ by construction, thus

$$\begin{aligned} & \rho^f(Y^1 + Y^2) = \int_0^\infty f(P\{Y^1 + Y^2 > x\}) dx \geq \int_0^\infty f([P\{X^1 + X^2 > x\} \vee d] - d) dx \\ \overset{(15)}{=} & \int_0^\infty g(P\{X^1 + X^2 > x\}) dx \overset{(16)}{=} \rho^g(X^1 + X^2), \end{aligned}$$

where we observe for (15) that $f((y \vee d) - d) = g(y)$ and for (16) that $X^1 + X^2 \geq 0$ by assumption. This shows (14).

Next, we show that the claim holds for $n + 1$ distortion functions, if it holds for up to n distortion functions. Assume that the induction hypothesis is true, and let $g^1, g^2, \ldots, g^{n+1}$ be distortion functions with parameters $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in [0, 1)$ and concave active parts. In this case,

$d = \sum_{i=1}^{n+1} \alpha_i$. The corresponding distortion risk measures are again denoted by $\rho^1, \rho^2, \dots, \rho^{n+1}$. Define

$$f^{(j)} = \min\{\widehat{g^1}, \widehat{g^2}, \dots, \widehat{g^j}\}, \quad d^{(j)} = \sum_{i=1}^{j} \alpha_i, \quad g^{(j)}(x) = \begin{cases} 0, & 0 \le x \le d^{(j)}, \\ f^{(j)}(x - d^{(j)}), & d^{(j)} < x \le 1 \end{cases}$$

$$(j = n, n+1)$$

Let $X^1, X^2, \dots, X^{n+1} \in L^\infty$ such that $\sum_{i=1}^{n+1} X^i = X$. Set

$$h(x) = \begin{cases} 0, & 0 \le x \le d, \\ f(x-d), & d < x \le 1, \end{cases}$$

with $f = \min\{\widehat{g^{(n)}}, \widehat{g^{n+1}}\}$. Then, using the induction hypothesis twice, we get that

$$\rho^h(X) \le \rho^{g^{(n)}}(\sum_{i=1}^{n} X^i) + \rho^{n+1}(X^{n+1}) \le \sum_{i=1}^{n+1} \rho^i(X^i). \tag{17}$$

Finally note that $d = d^{(n)} + \alpha_{n+1} = \sum_{i=1}^{n+1} \alpha_i$ and $f = \min\{\widehat{g^{(n)}}, \widehat{g^{n+1}}\} = \min\{\widehat{g^1}, \widehat{g^2}, \dots, \widehat{g^{n+1}}\}$, thus $h = g^{(n+1)}$. By Theorem 6 in Dhaene et al. (2012) we finally rewrite the left-hand side of equation (17) as

$$\rho^h(X) = \rho^{g^{(n+1)}}(X) = \int_{[0,1]} V@R_\lambda(X) g^{(n+1)}(d\lambda).$$

This proves the claim. □

A.4 Proof of Theorem 12

Proof. Due to the cash-invariance of risk measures, we may w.l.o.g suppose that $X \ge 0$. Renumbering the distortion functions and risk measures, we assume that $g^1(1-d+\alpha_1) < 1$. For $m \in \mathbb{N}$ we set

$$\begin{aligned} X^{1,m} &:= X \cdot \mathbf{1}_{\{X > V@R_{\alpha_1}(X)\}} + X \cdot \mathbf{1}_{\{V@R_d(X) \ge X\}} - m \cdot \mathbf{1}_{\{V@R_{\alpha_1}(X) \ge X > V@R_d(X)\}}, \\ X^{i,m} &:= (X+m) \cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{i-1}\alpha_l}(X) \ge X > V@R_{\sum_{l=1}^{i}\alpha_l}(X)\}} \qquad (i = 2, 3, \dots, n). \end{aligned}$$

Obviously, by construction $\sum_{i=1}^{n} X^{i,m} = X$.

Since g^i is a distortion function with parameter $\alpha_i > 0$ and $X \ge 0$, thus $X + m > 0$, we get

$$\rho^i(X^{i,m}) = 0, \qquad i = 2, 3, \dots, n.$$

We compute that

$$V@R_\lambda(X^{1,m}) = \begin{cases} V@R_\lambda(X), & \lambda < \alpha_1, \\ V@R_{\lambda+d-\alpha_1}(X), & \alpha_1 \le \lambda < 1 - d + \alpha_1, \\ -m, & 1 - d + \alpha_1 \le \lambda. \end{cases}$$

By Theorem 6 in Dhaene et al. (2012) we obtain

$$\rho^1(X^{1,m}) = \int_{[0,1]} V@R_\lambda(X^{1,m}) g^1(d\lambda) = c - m \cdot (1 - g^1(1 - d + \alpha_1))$$

where the constant $c \geq 0$ is given by

$$c = \int_{[0,1]} V@R_\lambda(X) \cdot \mathbf{1}_{[0,\alpha_1)}(\lambda) + V@R_{\lambda+d-\alpha_1}(X) \cdot \mathbf{1}_{[\alpha_1, 1-d+\alpha_1)}(\lambda) g^1(d\lambda) < \infty.$$

By assumption, $1 - g^1(1 - d + \alpha_1) > 0$, thus $\rho^1(X^{1,m}) \to -\infty$ as $m \to \infty$. □

A.5 Proof of Theorem 16

Proof. Using the notation of Theorem 8, we define for $i = 1, 2, \ldots, n$ the random variables $Z^i = Y \cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{i-1}\alpha_l}(Y) \geq Y > V@R_{\sum_{l=1}^{i}\alpha_l}(Y)\}}$. Then $Z^i \equiv 0$ on the set $\{V@R_{\alpha_i}(Z^i) \geq Z^i\}$ and $V@R_{\alpha_i}(Z^i) = 0$. Thus,

$$\begin{aligned}
\rho^i(X^i) &= \rho^i\left(Z^i + \frac{\text{essinf } X}{n}\right) = \rho^i(Z^i) + \frac{\text{essinf } X}{n} \\
&= \rho^i\left(Z^i \cdot \mathbf{1}_{\{V@R_{\alpha_i}(Z^i) \geq Z^i\}} + V@R_{\alpha_i}(Z^i) \cdot \mathbf{1}_{\{V@R_{\alpha_i}(Z^i) < Z^i\}}\right) + \frac{\text{essinf } X}{n} \\
&= \rho^i(0) + \frac{\text{essinf } X}{n}
\end{aligned}$$

This implies that $\sum_{i=1}^n \rho^i(X^i) = \sum_{i=1}^n \rho^i(0) + \text{essinf } X$. □

A.6 Proof of Theorem 20

Proof. Using the notation of Theorem 8, we define for $i = 1, 2, \ldots, n-1$ and $m > 0$ the random variables

$$X^{i,m} = (Y + m) \cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{i-1}\alpha_l}(Y) \geq Y > V@R_{\sum_{l=1}^{i}\alpha_l}(Y)\}} + \frac{\text{essinf } X}{n}$$

and

$$\begin{aligned}
X^{n,m} &= (Y + m) \cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{n-1}\alpha_l}(Y) \geq Y > V@R_d(Y)\}} + Y \cdot \mathbf{1}_{\{V@R_d(Y) \geq Y\}} + \frac{\text{essinf } X}{n} \\
X^{n+1,m} &= -m \cdot \mathbf{1}_{\{V@R_d(Y) < Y\}}.
\end{aligned}$$

Since $\rho^1, \rho^2, \ldots, \rho^n$ are V@R-type risk measures with parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$, we compute

$$\rho^i(X^{i,m}) = \rho^i(0) + \frac{\text{essinf } X}{n}, \quad i = 1, 2, \ldots, n-1,$$

$$\begin{aligned}
\rho^n(X^{n,m}) &= \rho^n\left(V@R_d(Y)\cdot \mathbf{1}_{\{V@R_{\sum_{l=1}^{n-1}\alpha_l}(Y)\geq Y > V@R_d(Y)\}} + Y\cdot \mathbf{1}_{\{V@R_d(Y)\geq Y\}} + \frac{\text{essinf } X}{n}\right) \\
&\leq \rho^n(0) + V@R_d(Y) + \frac{\text{essinf } X}{n}
\end{aligned}$$

Since ρ^{n+1} is strongly surplus sensitive at level d, we obtain that

$$\rho^{n+1}(X^{n+1,m}) \overset{m\to\infty}{\longrightarrow} -\infty.$$

Thus,

$$\sum_{i=1}^{n+1}\rho^i(X^{i,m}) \;\leq\; \sum_{i=1}^{n}\rho^i(0) + V@R_d(Y) + \text{ essinf } X + \rho^{n+1}(X^{n+1,m}) \;\overset{m\to\infty}{\longrightarrow}\; -\infty.$$

$\square$

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