

# EXISTENCE OF LÉVY TERM STRUCTURE MODELS

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**ABSTRACT.** Lévy driven term structure models have become an important subject in the mathematical finance literature. This paper provides a comprehensive analysis of the Lévy driven Heath–Jarrow–Morton type term structure equation. This includes a full proof of existence and uniqueness in particular, which seems to have been lacking in the finance literature so far.

**Key Words:** forward curve spaces; Lévy term structure models, stochastic integration in Hilbert spaces; strong, weak and mild solutions of infinite dimensional SDE's,

## 1. INTRODUCTION

A zero coupon bond with maturity  $T$  is a financial asset which pays the holder one unit of cash at  $T$ . Its price at  $t \leq T$  can be written as

$$P(t, T) = \exp\left(-\int_t^T f(t, u) du\right)$$

where  $f(t, T)$  is the forward rate for date  $T$ . The classical continuous framework for the evolution of the forward rates goes back to Heath, Jarrow and Morton (HJM) [32]. They assume that, under the risk-neutral measure, for every date  $T$ , the forward rates  $f(t, T)$  follow an Itô process of the form

$$(1.1) \quad df(t, T) = \left(\sum_{i=1}^n \sigma_i(t, T) \int_t^T \sigma_i(t, s) ds\right) dt + \sum_{i=1}^n \sigma_i(t, T) dW_t^i, \quad t \in [0, T],$$

where  $W = (W^1, \dots, W^n)$  is a standard Brownian motion in  $\mathbb{R}^n$ . The dynamics (1.1) guarantee that the discounted zero coupon bond price processes

$$e^{-\int_0^t f(s, s) ds} P(t, T), \quad t \in [0, T],$$

are local martingales for all maturities  $T$ . This is the well known condition for the absence of arbitrage in the bond market model.

Empirical studies have revealed that models based on Brownian motion only provide a poor fit to observed market data. We refer to [51, Chap. 5], where it is argued that empirically observed log returns of zero coupon bonds are not normally distributed, a fact, which has long before been known for the distributions of stock returns. Björk et al. [5, 6], Eberlein et al. [22, 21, 16, 19, 20, 18] and others ([55, 37, 33]) thus proposed to replace the classical Brownian motion  $W$  in (1.1) by a more general process  $X = (X^1, \dots, X^n)$ , also taking into account the occurrence of jumps. If  $X$  is a Lévy process, this leads to

$$(1.2) \quad df(t, T) = \alpha_{\text{HJM}}(t, T) dt + \sum_{i=1}^n \sigma_i(t, T) dX_t^i, \quad t \in [0, T].$$

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The HJM drift in (1.1) accordingly is replaced by some appropriate  $\alpha_{\text{HJM}}(t, T)$ , which is determined by  $\sigma(t, T)$  and the cumulant generating function of  $X$ , see (2.4) below.

Equation (1.2) constitutes a generic description of the forward rate process  $f(t, T, \omega)$  in terms of a stochastic volatility process  $\sigma(t, T, \omega)$ . From a financial modelling point of view one would rather consider  $\sigma(t, T, \omega)$ , and thus  $\alpha_{\text{HJM}}(t, T, \omega)$ , to be a function of the prevailing forward curve  $T \mapsto f(t-, T, \omega) = \lim_{s \uparrow t} f(s, T, \omega)$ , that is

$$\sigma(t, T, \omega) = \sigma(t, T, f(t-, \cdot, \omega)), \quad \alpha_{\text{HJM}}(t, T, \omega) = \alpha_{\text{HJM}}(t, T, f(t, \cdot, \omega)).$$

This makes  $f(t, T)$  being a *solution* of the stochastic equation

$$(1.3) \quad \begin{cases} df(t, T) &= \alpha_{\text{HJM}}(t, T, f(t, \cdot))dt + \sum_{i=1}^n \sigma_i(t, T, f(t-, \cdot))dX_t^i, \quad t \in [0, T], \\ f(0, T) &= h_0(T) \end{cases}$$

for some given initial forward curve  $h_0(T)$ .

*Term structure models* of the type (1.3) are frequently considered in the literature. The typical assumption is that drift  $\alpha_{\text{HJM}}$  and volatility  $\sigma$  depend on the current state of the short rate,  $\sigma(t, T, \omega) = \sigma(t, T, f(t-, t, \omega))$ , as in [38], [52], [4], [34] and [33] (the latter studies models driven by jump-diffusions). A model, where the volatility  $\sigma$  is allowed to depend on a finite number of benchmark forward rates, is considered in [11] and [12]. We emphasize that these papers, whose setups are special cases of our present framework, *assume* that the forward rates  $f(t, T)$  evolve according to an equation of the kind (1.3). To our knowledge, there has not been yet an explicit proof for the *existence* of a solution to (1.3) in the mathematical finance literature. We thus provide such a proof in our paper (Theorem 4.6 and Corollary 4.7).

Note that (1.3) is an infinite-dimensional and therefore non-trivial problem. In fact, (1.3) is not simply a system of infinitely many univariate stochastic equations for  $f(t, T)$ ,  $t \in [0, T]$ , indexed by  $T$ . Indeed, these equations are coupled as  $\alpha_{\text{HJM}}$  and  $\sigma$  depend on the entire forward curve  $f(t-, \cdot)$ , say e.g. on the short rate  $f(t-, t)$ , which is a functional of  $f(t-, \cdot)$ . To express this functional dependence, one switches best to the alternative parametrization

$$r_t(x) = f(t, t+x), \quad x \geq 0,$$

which is due to Musiela [46]. We then write  $S_t f(x) := f(x+t)$  for the shift operator  $S_t$ . Equation (1.3) becomes in integrated form

$$(1.4) \quad r_t(x) = S_t h_0(x) + \int_0^t S_{t-s} \alpha_{\text{HJM}}(s, s+x, r_s) ds + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, s+x, r_{s-}) dX_s^i,$$

where  $S_{t-s}$  operates on the functions  $x \mapsto \alpha_{\text{HJM}}(s, s+x, r_s)$  and  $x \mapsto \sigma_i(s, s+x, r_{s-})$ . Hence, in the spirit of Da Prato and Zabczyk [13], the process  $r_t$  is a so called *mild solution* of the stochastic differential equation

$$(1.5) \quad \begin{cases} dr_t &= \left( \frac{d}{dx} r_t + \alpha_{\text{HJM}}(t, r_t) \right) dt + \sum_{i=1}^n \sigma_i(t, r_{t-}) dX_t^i, \\ r_0 &= h_0 \end{cases}$$

in some appropriate Hilbert space  $H$  of forward curves, where  $\frac{d}{dx}$  becomes the generator of the strongly continuous semigroup of shifts  $S_t$ . Note the slight abuse of notation  $\alpha_{\text{HJM}}(t, t+\cdot, r) \rightsquigarrow \alpha_{\text{HJM}}(t, r)$  and  $\sigma_i(t, t+\cdot, r) \rightsquigarrow \sigma_i(t, r)$ . In the sequel, we are therefore concerned with the Lévy HJMM (Heath–Jarrow–Morton–Musiela) equation (1.5) in various choices of the state space  $H$ .

Several authors have dealt with the existence issue for (1.5) for the Brownian motion case  $X = W$ . Björk and Svensson [7] chose the state space  $H_{\beta,\gamma}$  small enough such that  $\frac{d}{dx} : H_{\beta,\gamma} \rightarrow H_{\beta,\gamma}$  becomes a bounded linear operator. In this case, the methods from finite dimension essentially carry over to (1.5). It turns out, however, that the Björk–Svensson space  $H_{\beta,\gamma}$  is too small and does not contain some important classical term structure models (see [25]). In [24], we thus analyzed and solved (1.5) for  $X = W$  on a larger space  $H_w$ , where  $\frac{d}{dx}$  becomes unbounded.

In this paper we provide the existence proof for (1.5) for the Lévy case. We proceed as follows. Using an existence result for general Hilbert space valued stochastic differential equations from the appendix, we first show existence for (1.5) in the Björk–Svensson space  $H_{\beta,\gamma}$ . However, often it turns out that  $H_{\beta,\gamma}$  is too small to assert that  $\alpha_{\text{HJM}}$  lies in  $H_{\beta,\gamma}$ , even for the very simple case where  $\sigma$  is constant and the driver  $X$  is a compound Poisson process (Example 3.4). Afterwards, we thus consider (1.5) in the larger state space  $H_w$  from [24] where  $\frac{d}{dx}$  becomes unbounded.

Term structure models based on *infinite dimensional* driving processes  $X$  are discussed e.g. in [36] and [47] for the Lévy case. Again, in these papers it is typically assumed that the forward curve evolution satisfies a stochastic differential equation, but the authors do not treat existence and uniqueness of solutions.

The remainder of the paper is organized as follows. In Section 2 we introduce some notation and specify the HJM drift  $\alpha_{\text{HJM}}$ , which ensures that the bond market is free of arbitrage. In Section 3 we treat the existence of strong solutions to (1.5) on the Björk–Svensson space  $H_{\beta,\gamma}$ . Afterwards, Section 4 is devoted to the existence of mild and weak solutions to (1.5) on the larger space  $H_w$  where  $\frac{d}{dx}$  becomes unbounded. Section 5 concludes.

For our results of Section 3 and Section 4 we apply an existence result for general Hilbert space valued stochastic differential equations, which is derived in the appendix. The ground for this result, Theorem C.1, is prepared by two works of van Gaans [26, 27]. In addition to his result [27, Thm. 4.1] we prove that the mild solution to (1.5) has a càdlàg modification, and that there exists a unique weak solution.

The càdlàg property of the solution is an important feature for financial applications. Indeed, general arbitrage theory [15] requires that the basic financial instruments, here the implied zero coupon bond prices  $P(t, T) = \exp(-\int_0^{T-t} r_t(x) dx)$ , are real semimartingales and therefore have càdlàg paths. This essentially requires càdlàg paths of the weak solution  $(r_t)$ , which is satisfied in our framework.

As it turns out, the stochastic integral of van Gaans [27] is not consistent with the usual Itô-integral, which is used for financial modelling. Therefore, after giving an overview and the required notation in Appendix A, we show in Appendix B that the stochastic integral of van Gaans always has a càdlàg modification and analyze when it coincides with the Itô-integral. Then, in Appendix C, we prove Theorem C.1, the existence and uniqueness result for Hilbert space valued stochastic equations. At the end of Appendix C we give an overview of related literature.

## 2. THE HJM DRIFT CONDITION

Throughout this text,  $X^1, \dots, X^n$  denote independent real-valued Lévy processes on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the usual conditions.

Let  $H$  be a separable Hilbert space representing the space of forward curves and let  $\sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H$  be the volatilities. Recall that a Lévy term structure model of the form (1.5) is free of arbitrage if the probability measure  $\mathbb{P}$  is a local martingale measure, that is all discounted bond prices are local martingales.

In order to provide a condition which ensures that  $\mathbb{P}$  is a local martingale measure, we assume that there are compact intervals  $[a_1, b_1], \dots, [a_n, b_n]$  having zero as an inner point such that the Lévy measures  $F_1, \dots, F_n$  of  $X^1, \dots, X^n$ , respectively, satisfy for  $i = 1, \dots, n$

$$(2.1) \quad \int_{|x|>1} e^{zx} F_i(dx) < \infty, \quad z \in [a_i, b_i].$$

Condition (2.1) ensures that the cumulant generating functions

$$(2.2) \quad \Psi_i(z) := \ln \mathbb{E}[\exp(zX_1^i)], \quad i = 1, \dots, n$$

exist on  $[a_i, b_i]$  and that they are of class  $C^\infty$  (see [54, Lemma 26.4]). Moreover, the Lévy processes  $X^i$  possess moments of arbitrary order. Let  $[c_i, d_i] \subset (a_i, b_i)$  be further compact intervals having zero as an inner point.

For any continuous function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  we define  $Th : \mathbb{R}_+ \rightarrow \mathbb{R}$  as

$$(2.3) \quad Th(x) := \int_0^x h(\eta) d\eta.$$

For  $i = 1, \dots, n$  denote

$$A_H^{\Psi_i} := \{h \in H : -Th(\mathbb{R}_+) \subset [c_i, d_i]\}.$$

Provided  $\sigma_i(\mathbb{R}_+ \times H) \subset A_H^{\Psi_i}$  for  $i = 1, \dots, n$ , the *HJM drift*

$$(2.4) \quad \begin{aligned} \alpha_{\text{HJM}}(t, r)(x) &= \sum_{i=1}^n \frac{d}{dx} \Psi_i \left( - \int_0^x \sigma_i(t, r)(\eta) d\eta \right) \\ &= - \sum_{i=1}^n \sigma_i(t, r)(x) \Psi_i' \left( - \int_0^x \sigma_i(t, r)(\eta) d\eta \right) \end{aligned}$$

is well defined pointwise for all  $x$ . The HJM drift condition (2.4) implies that  $\mathbb{P}$  is a local martingale measure. It is derived in [21, Sec. 2.1] for the present Lévy case, using the results of the more general setup in [5]. For an analogous drift condition in the infinite dimensional Lévy setting, see [36].

The HJM drift specification (2.4) causes some problems for an immediate application of Theorem C.1. First of all, we have to ensure that  $\alpha_{\text{HJM}}(t, r) \in H$  for all  $(t, r) \in \mathbb{R}_+ \times H$ . Furthermore, we have to establish for an application of Theorem C.1 that for Lipschitz functions  $\sigma_1, \dots, \sigma_n$  the drift  $\alpha_{\text{HJM}}$  is again a Lipschitz function.

These demandings emphasize that we have to be careful about the choice of the space  $H$  of forward curves. Another desirable feature of  $H$  is that for every  $x \in \mathbb{R}_+$  the point evaluation  $h \mapsto h(x) : H \rightarrow \mathbb{R}$  is a continuous linear functional. Because then the variation of constants formula (1.4) is satisfied for all  $x \in \mathbb{R}_+$ , whenever  $(r_t)$  is a mild solution of (1.5).

In the upcoming Section 3 we deal with the existence of strong solutions to (1.5), and Section 4 is devoted to the existence of mild and weak solutions to (1.5).

### 3. FORWARD CURVE EVOLUTIONS AS STRONG SOLUTIONS OF INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, where we deal with the existence of strong solutions to (1.5), we consider the spaces  $H_{\beta, \gamma}$  of forward curves, which have been used by Björk and Svensson in [7].

We fix real numbers  $\beta > 1$  and  $\gamma > 0$ . Let  $H_{\beta,\gamma}$  be the linear space of all  $h \in C^\infty(\mathbb{R}_+; \mathbb{R})$  satisfying

$$\sum_{n=0}^{\infty} \left(\frac{1}{\beta}\right)^n \int_0^\infty \left(\frac{d^n h(x)}{dx^n}\right)^2 e^{-\gamma x} dx < \infty,$$

We define the inner product

$$\langle g, h \rangle_{\beta,\gamma} := \sum_{n=0}^{\infty} \left(\frac{1}{\beta}\right)^n \int_0^\infty \left(\frac{d^n g(x)}{dx^n}\right) \left(\frac{d^n h(x)}{dx^n}\right) e^{-\gamma x} dx$$

and denote the corresponding norm by  $\|\cdot\|_{\beta,\gamma}$ .

**3.1. Proposition.** *The space  $(H_{\beta,\gamma}, \langle \cdot, \cdot \rangle_{\beta,\gamma})$  is a separable Hilbert space and for each  $x \in \mathbb{R}_+$ , the point evaluation  $h \mapsto h(x) : H_{\beta,\gamma} \rightarrow \mathbb{R}$  is a continuous linear functional.*

*Proof.* This is a consequence of [7, Prop. 4.2].  $\square$

The fact that each point evaluation is a continuous linear functional ensures that forward curves  $(r_t)$  solving (1.5) satisfy the variation of constants formula (1.4).

**3.2. Proposition.** *We have  $\frac{d}{dx} \in \mathcal{L}(H_{\beta,\gamma})$ , i.e.  $\frac{d}{dx}$  is a bounded linear operator on  $H_{\beta,\gamma}$ .*

*Proof.* The assertion is a consequence of [7, Prop. 4.2].  $\square$

**3.3. Theorem.** *Let  $\sigma_i : \mathbb{R}_+ \times H_{\beta,\gamma} \rightarrow H_{\beta,\gamma}$  be continuous and satisfying  $\sigma_i(\mathbb{R}_+ \times H_{\beta,\gamma}) \subset A_{H_{\beta,\gamma}}^{\Psi_i}$  for  $i = 1, \dots, n$ . Assume that  $\alpha_{\text{HJM}}(t, r) \in H_{\beta,\gamma}$  for all  $(t, r) \in \mathbb{R}_+ \times H_{\beta,\gamma}$ . Furthermore, assume that  $\alpha_{\text{HJM}}(t, r) : \mathbb{R}_+ \times H_{\beta,\gamma} \rightarrow H_{\beta,\gamma}$  is continuous and that there is a constant  $L \geq 0$  such that for all  $t \in \mathbb{R}_+$  and  $h_1, h_2 \in H_{\beta,\gamma}$  we have*

$$\begin{aligned} \|\alpha_{\text{HJM}}(t, h_1) - \alpha_{\text{HJM}}(t, h_2)\|_{\beta,\gamma} &\leq L \|h_1 - h_2\|_{\beta,\gamma}, \\ \|\sigma_i(t, h_1) - \sigma_i(t, h_2)\|_{\beta,\gamma} &\leq L \|h_1 - h_2\|_{\beta,\gamma}, \quad i = 1, \dots, n. \end{aligned}$$

*Then, for each  $h_0 \in H_{\beta,\gamma}$ , there exists a unique strong adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (1.5) with  $r_0 = h_0$  satisfying*

$$(3.1) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|r_t\|_{\beta,\gamma}^2 \right] < \infty \quad \text{for all } T > 0.$$

*Proof.* Taking into account Proposition 3.2, the result is a consequence of Corollary C.2.  $\square$

Unfortunately, Theorem 3.3 has some shortcomings, namely it is demanded that the drift term  $\alpha_{\text{HJM}}$  according to the HJM drift condition (2.4) maps again into the space  $H_{\beta,\gamma}$ . The following simple counter example shows that this condition may be violated.

**3.4. Example.** *Let  $\sigma = -1$  and  $X$  be a compound Poisson process with intensity  $\lambda = 1$  and jump size distribution  $N(0, 1)$ . Notice that the compound Poisson process satisfies the exponential moments condition (2.1) for all  $z \in \mathbb{R}$ , because its Lévy measure is given by*

$$F(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

But we have  $\alpha_{\text{HJM}} \notin H_{\beta,\gamma}$ , because

$$\begin{aligned} \int_0^\infty \alpha_{\text{HJM}}(x)^2 e^{-\gamma x} dx &= \int_0^\infty \left( \frac{d}{dx} \Psi(x) \right)^2 e^{-\gamma x} dx \\ &= \int_0^\infty \left( \frac{d}{dx} \left( e^{\frac{x^2}{2}} - 1 \right) \right)^2 e^{-\gamma x} dx = \int_0^\infty x^2 e^{x^2 - \gamma x} dx = \infty. \end{aligned}$$

The phenomena that the drift  $\alpha_{\text{HJM}}$  may be located outside the space of forward curves  $H_{\beta,\gamma}$  has to do with the fact that the space  $H_{\beta,\gamma}$  is a very small space in a sense, in particular, every function must necessarily be real-analytic (see [7, Prop. 4.2]).

The small size of this space arises from the requirement that  $\frac{d}{dx}$  should be a bounded operator, because we deal with the existence of *strong* solutions. When dealing with *mild* and *weak* solutions in the next Section 4, problems of this kind will not occur.

Nevertheless, for certain types of term structure models, we can apply Theorem 3.3. For this purpose, we proceed with a lemma. For a given real-analytic function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  it is, in general, difficult to decide whether  $h$  belongs to  $H_{\beta,\gamma}$  or not. For the following functions this can be provided.

**3.5. Lemma.** *Every polynomial  $p$  belongs to  $H_{\beta,\gamma}$ , and for  $\delta \in \mathbb{R}$  satisfying  $\delta^2 < \beta$  and  $\delta < \frac{\gamma}{2}$ , the function  $h(x) = e^{\delta x}$  belongs to  $H_{\beta,\gamma}$ .*

*Proof.* The first statement is clear. For  $h(x) = e^{\delta x}$  we obtain

$$\begin{aligned} \sum_{n=0}^\infty \left( \frac{1}{\beta} \right)^n \int_0^\infty \left( \frac{d^n h(x)}{dx^n} \right)^2 e^{-\gamma x} dx &= \sum_{n=0}^\infty \left( \frac{1}{\beta} \right)^n \int_0^\infty (\delta^n e^{\delta x})^2 e^{-\gamma x} dx \\ &= \sum_{n=0}^\infty \left( \frac{\delta^2}{\beta} \right)^n \int_0^\infty e^{-(\gamma - 2\delta)x} dx = \frac{1}{1 - \frac{\delta^2}{\beta}} \cdot \frac{1}{\gamma - 2\delta} = \frac{\beta}{(\beta - \delta^2)(\gamma - 2\delta)}, \end{aligned}$$

whence  $h \in H_{\beta,\gamma}$ . □

Let  $n = 3$ , that is we have three independent driving processes. We denote by  $X^1, X^2$  two standard Wiener processes, and  $X^3$  is a Poisson process with intensity  $\lambda > 0$ . We specify the volatilities as

$$\sigma_1(r)(x) = \varphi_1(r)p(x), \quad \sigma_2(r)(x) = \varphi_2(r)e^{\delta x} \quad \text{and} \quad \sigma_3(r)(x) = -\eta,$$

where  $p$  is a polynomial,  $\delta, \eta \in \mathbb{R}$  satisfy  $4\delta^2 < \beta$ ,  $\delta < \frac{\gamma}{4}$  and  $\eta^2 < \beta$ ,  $\eta < \frac{\gamma}{2}$ , and where  $\varphi_i : H_{\gamma,\beta} \rightarrow \mathbb{R}$  for  $i = 1, 2$ . Note that  $\sigma_i(H_{\beta,\gamma}) \subset H_{\beta,\gamma}$  for  $i = 1, 2, 3$  by Lemma 3.5. The drift according to the HJM drift condition (2.4) is given by

$$\alpha_{\text{HJM}}(r)(x) = \frac{d}{dx} \left[ \frac{1}{2} \varphi_1(r)^2 q(x)^2 + \frac{1}{2} \varphi_2(r)^2 \left( \frac{e^{\delta x} - 1}{\delta} \right)^2 + \lambda (e^{\eta x} - 1) \right],$$

where  $q(x) = \int_0^x p(\eta) d\eta$  is again a polynomial. From Lemma 3.5 and Proposition 3.2 we infer  $\alpha_{\text{HJM}}(H_{\beta,\gamma}) \subset H_{\beta,\gamma}$ .

**3.6. Proposition.** *Assume there is a constant  $L \geq 0$  such that for all  $h_1, h_2 \in H_{\beta,\gamma}$  we have*

$$\begin{aligned} |\varphi_i(h_1) - \varphi_i(h_2)| &\leq L \|h_1 - h_2\|_{\beta,\gamma}, \quad i = 1, 2 \\ |\varphi_i(h_1)^2 - \varphi_i(h_2)^2| &\leq L \|h_1 - h_2\|_{\beta,\gamma}, \quad i = 1, 2. \end{aligned}$$

*Then, for each  $h_0 \in H_{\beta,\gamma}$ , there exists a unique strong adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (1.5) with  $r_0 = h_0$  satisfying (3.1).*

*Proof.* We have for all  $h_1, h_2 \in H_{\beta, \gamma}$

$$\begin{aligned}\|\sigma_1(h_1) - \sigma_1(h_2)\| &\leq L\|p\|_{\beta, \gamma}\|h_1 - h_2\|_{\beta, \gamma}, \\ \|\sigma_2(h_1) - \sigma_2(h_2)\| &\leq L\|e^{\delta \bullet}\|_{\beta, \gamma}\|h_1 - h_2\|_{\beta, \gamma}.\end{aligned}$$

Using Proposition 3.2, we obtain for all  $h_1, h_2 \in H_{\beta, \gamma}$

$$\|\alpha_{\text{HJM}}(h_1) - \alpha_{\text{HJM}}(h_2)\|_{\beta, \gamma} \leq \frac{L}{2}\|A\|_{\mathcal{L}(H_{\beta, \gamma})} (\|q^2\|_{\beta, \gamma} + \|\frac{1}{\delta^2}(e^{\delta \bullet} - 1)^2\|_{\beta, \gamma}) \|h_1 - h_2\|_{\beta, \gamma}.$$

Applying Theorem 3.3 completes the proof.  $\square$

In order to generalize Proposition 3.6, by allowing that  $\eta$  may depend on the present state of the forward curve, we prepare two auxiliary results.

**3.7. Lemma.** *Let  $\gamma > 0$  and  $g, h \in C^1(\mathbb{R}_+; \mathbb{R})$ . Assume there are  $c > 0$ ,  $\varepsilon \in (-\infty, \gamma)$  and  $x_0 \in \mathbb{R}_+$  such that*

$$|g(x)h(x)| \leq ce^{\varepsilon x} \quad \text{for all } x \geq x_0.$$

*Then we have*

$$\begin{aligned}&\int_0^\infty g(x)h(x)e^{-\gamma x} dx \\ &= \frac{1}{\gamma} \left[ g(0)h(0) + \int_0^\infty g'(x)h(x)e^{-\gamma x} dx + \int_0^\infty g(x)h'(x)e^{-\gamma x} dx \right].\end{aligned}$$

*Proof.* Performing partial integration with three factors, we obtain

$$\begin{aligned}\left[ g(x)h(x)e^{-\gamma x} \right]_0^\infty &= \int_0^\infty g'(x)h(x)e^{-\gamma x} dx + \int_0^\infty g(x)h'(x)e^{-\gamma x} dx \\ &\quad - \gamma \int_0^\infty g(x)h(x)e^{-\gamma x} dx.\end{aligned}$$

By hypothesis, we have  $\lim_{x \rightarrow \infty} g(x)h(x)e^{-\gamma x} = 0$ , and so the stated formula follows.  $\square$

**3.8. Lemma.** *Let  $\gamma > 0$  and  $h \in C^2(\mathbb{R}_+; \mathbb{R})$  be such that  $h, h', h'' \geq 0$ . Assume there are  $c > 0$ ,  $\varepsilon \in (-\infty, \frac{\gamma}{2})$  and  $x_0 \in \mathbb{R}_+$  such that*

$$|h(x)| \leq ce^{\varepsilon x} \quad \text{and} \quad |h'(x)| \leq ce^{\varepsilon x} \quad \text{for all } x \geq x_0.$$

*Then we have*

$$\int_0^\infty h'(x)^2 e^{-\gamma x} dx \leq \frac{\gamma^2}{2} \int_0^\infty h(x)^2 e^{-\gamma x} dx.$$

*Proof.* Using two times Lemma 3.7, we obtain

$$\begin{aligned}\int_0^\infty h(x)^2 e^{-\gamma x} dx &= \frac{2}{\gamma} \int_0^\infty h(x)h'(x)e^{-\gamma x} dx + \frac{1}{\gamma} h(0)^2 \\ &= \frac{2}{\gamma^2} \left[ \int_0^\infty h'(x)^2 e^{-\gamma x} dx + \int_0^\infty h(x)h''(x)e^{-\gamma x} dx \right] + \frac{1}{\gamma} \left[ h(0)^2 + \frac{2}{\gamma} h(0)h'(0) \right].\end{aligned}$$

Since  $h, h', h'' \geq 0$  by hypothesis, the stated inequality follows.  $\square$

Now we generalize Proposition 3.6 by assuming that  $\eta : H_{\beta, \gamma} \rightarrow \mathbb{R}$  is allowed to depend on the current state of the forward curve. The rest of our present framework is exactly as in Proposition 3.6.

**3.9. Proposition.** *Assume that, in addition to the hypothesis of Proposition 3.6, we have  $\gamma \leq \sqrt{2}$ ,  $\eta(H_{\beta, \gamma}) \subset [0, \frac{\gamma}{2}] \cap [0, \sqrt{\beta})$  and*

$$|\eta(h_1) - \eta(h_2)| \leq L\|h_1 - h_2\|_{\beta, \gamma}$$

*for all  $h_1, h_2 \in H_{\beta, \gamma}$ . Then, for each  $h_0 \in H_{\beta, \gamma}$ , there exists a unique strong adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (1.5) with  $r_0 = h_0$  satisfying (3.1).*

*Proof.* It suffices to show that  $\Gamma : H_{\beta,\gamma} \rightarrow H_{\beta,\gamma}$  defined as  $\Gamma(r)(x) := e^{\eta(r)x}$  is Lipschitz continuous. So let  $h_1, h_2 \in H_{\beta,\gamma}$  be arbitrary. Without loss of generality we assume that  $\eta(h_2) \leq \eta(h_1)$ . Observe that all derivatives of  $\Gamma(h_1) - \Gamma(h_2)$  are non-negative. So we obtain by applying Lemma 3.8 (notice that  $\gamma \leq \sqrt{2}$  by hypothesis), and the Lipschitz property  $|e^x - e^y| \leq e^x|x - y|$  for  $y \leq x$  that

$$\begin{aligned} \|\Gamma(h_1) - \Gamma(h_2)\|_{\beta,\gamma}^2 &= \sum_{n=0}^{\infty} \left(\frac{1}{\beta}\right)^n \int_0^{\infty} \left(\eta(h_1)^n e^{\eta(h_1)x} - \eta(h_2)^n e^{\eta(h_2)x}\right)^2 e^{-\gamma x} dx \\ &\leq \frac{\beta}{\beta-1} \int_0^{\infty} \left(e^{\eta(h_1)x} - e^{\eta(h_2)x}\right)^2 e^{-\gamma x} dx \\ &\leq \frac{\beta}{\beta-1} \int_0^{\infty} \left(e^{\eta(h_1)x} (\eta(h_1) - \eta(h_2))x\right)^2 e^{-\gamma x} dx \\ &\leq \frac{\beta}{\beta-1} \left(\int_0^{\infty} \left(xe^{\eta(h_1)x}\right)^2 e^{-\gamma x} dx\right) L^2 \|h_1 - h_2\|_{\beta,\gamma}^2. \end{aligned}$$

The integral is finite, because we have  $\eta(h_1) \in [0, \frac{\gamma}{2})$  by assumption. Applying Theorem 3.3 finishes the proof.  $\square$

#### 4. FORWARD CURVE EVOLUTIONS AS MILD AND WEAK SOLUTIONS OF INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, where we deal with the existence of mild and weak solutions to (1.5), we consider the spaces  $H_w$  of forward curves, which have been introduced in [24, Chap. 5].

Let  $w : \mathbb{R}_+ \rightarrow [1, \infty)$  be a non-decreasing  $C^1$ -function such that  $w^{-\frac{1}{3}} \in L^1(\mathbb{R}_+)$ .

4.1. **Example.**  $w(x) = e^{\alpha x}$ , for  $\alpha > 0$ .

4.2. **Example.**  $w(x) = (1+x)^\alpha$ , for  $\alpha > 3$ .

Let  $H_w$  be the linear space of all absolutely continuous functions  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying

$$\int_{\mathbb{R}_+} |h'(x)|^2 w(x) dx < \infty,$$

where  $h'$  denotes the weak derivative of  $h$ . We define the inner product

$$(g, h)_w := g(0)h(0) + \int_{\mathbb{R}_+} g'(x)h'(x)w(x) dx$$

and denote the corresponding norm by  $\|\cdot\|_w$ . Since forward curves flatten for large time to maturity  $x$ , the choice of  $H_w$  is reasonable from an economic point of view.

4.3. **Proposition.** *The space  $(H_w, (\cdot, \cdot)_w)$  is a separable Hilbert space. Each  $h \in H_w$  is continuous, bounded and the limit  $h(\infty) := \lim_{x \rightarrow \infty} h(x)$  exists. Moreover, for each  $x \in \mathbb{R}_+$ , the point evaluation  $h \mapsto h(x) : H_w \rightarrow \mathbb{R}$  is a continuous linear functional.*

*Proof.* All of these statements can be found in the proof of [24, Thm. 5.1.1].  $\square$

The fact that each point evaluation is a continuous linear functional ensures that forward curves  $(r_t)$  solving (1.5) satisfy the variation of constants formula (1.4).

Defining the constants  $C_1, \dots, C_4 > 0$  as

$$C_1 := \|w^{-1}\|_{L^1(\mathbb{R}_+)}^{\frac{1}{2}}, \quad C_2 := 1 + C_1, \quad C_3 := \|w^{-\frac{1}{3}}\|_{L^1(\mathbb{R}_+)}^2, \quad C_4 := \|w^{-\frac{1}{3}}\|_{L^1(\mathbb{R}_+)}^{\frac{7}{2}},$$



we have for all  $h \in H_w$  the estimates

$$(4.1) \quad \|h'\|_{L^1(\mathbb{R}_+)} \leq C_1 \| \|h\|_w,$$

$$(4.2) \quad \|h\|_{L^\infty(\mathbb{R}_+)} \leq C_2 \| \|h\|_w,$$

$$(4.3) \quad \|h - h(\infty)\|_{L^1(\mathbb{R}_+)} \leq C_3 \| \|h\|_w,$$

$$(4.4) \quad \|(h - h(\infty))^4 w\|_{L^1(\mathbb{R}_+)} \leq C_4 \| \|h\|_w^4,$$

which also follows by inspecting the proof of [24, Thm. 5.1.1].

Since for an application of Theorem C.1 we require that the shift semigroup  $(S_t)_{t \geq 0}$  defined by  $S_t h = h(t + \cdot)$  for  $t \in \mathbb{R}_+$  is pseudo-contractive in a closed subspace of  $H_w$ , we perform an idea, which is due to Tehranchi [57], namely we change to the inner product

$$\langle g, h \rangle_w := g(\infty)h(\infty) + \int_{\mathbb{R}_+} g'(x)h'(x)w(x)dx$$

and denote the corresponding norm by  $\| \cdot \|_w$ . The estimates (4.1)–(4.4) are also valid with the norm  $\| \cdot \|_w$  for all  $h \in H_w$ , which is proven exactly as for the original norm  $\| \| \cdot \|_w$ . Therefore we conclude, by using (4.2),

$$\frac{1}{(1 + C_2^2)^{\frac{1}{2}}} \| \|h\|_w \leq \| \|h\|_w \leq (1 + C_2^2)^{\frac{1}{2}} \| \|h\|_w, \quad h \in H_w$$

showing that  $\| \cdot \|_w$  and  $\| \| \cdot \|_w$  are equivalent norms on  $H_w$ . From now on, we shall work with the norm  $\| \cdot \|_w$ .

**4.4. Proposition.**  $(S_t)$  is a  $C_0$ -semigroup in  $H_w$  with generator  $\frac{d}{dx} : \mathcal{D}(\frac{d}{dx}) \subset H_w \rightarrow H_w$ ,  $\frac{d}{dx}h = h'$ , and domain

$$\mathcal{D}(\frac{d}{dx}) = \{h \in H_w \mid h' \in H_w\}.$$

The subspace  $H_w^0 := \{h \in H_w \mid h(\infty) = 0\}$  is a closed subspace of  $H_w$  and  $(S_t)$  is contractive in  $H_w^0$  with respect to the norm  $\| \cdot \|_w$ .

*Proof.* Except for the last statement, we refer to the proof of [24, Thm. 5.1.1]. By the monotonicity of  $w$  we have

$$\|S_t h\|_w^2 = \int_{\mathbb{R}_+} |h'(x+t)|^2 w(x)dx \leq \| \|h\|_w^2$$

for all  $t \in \mathbb{R}_+$  and  $h \in H_w^0$ , showing that  $(S_t)$  is contractive in  $H_w^0$ .  $\square$

We define for any  $h = (h_1, \dots, h_n) \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$

$$(4.5) \quad \Sigma h(x) := - \sum_{i=1}^n h_i(x) \Psi_i' \left( - \int_0^x h_i(\eta) d\eta \right), \quad x \in \mathbb{R}_+.$$

**4.5. Proposition.** There is a constant  $C_5 > 0$  such that for all  $g, h \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$  we have

$$(4.6) \quad \|\Sigma g - \Sigma h\|_w \leq C_5 \sum_{i=1}^n (1 + \| \|h_i\|_w + \| \|g_i\|_w + \| \|g_i\|_w^2) \| \|g_i - h_i\|_w.$$

Furthermore, for each  $h \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$  we have  $\Sigma h \in H_w^0$ , and the map  $\Sigma : \Pi_{i=1}^n A_{H_w^0}^{\Psi_i} \rightarrow H_w^0$  is continuous.

*Proof.* We define

$$K_i := \sup_{x \in [c_i, d_i]} |\Psi_i'(x)|, \quad L_i := \sup_{x \in [c_i, d_i]} |\Psi_i''(x)| \quad \text{and} \quad M_i := \sup_{x \in [c_i, d_i]} |\Psi_i'''(x)|$$

for  $i = 1, \dots, n$ . By the boundedness of the derivatives  $\Psi'_i$  on  $[c_i, d_i]$ , the definition (4.5) of  $\Sigma$  yields that for each  $h \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$  the limit  $\Sigma h(\infty) := \lim_{x \rightarrow \infty} \Sigma h(x)$  exists and

$$(4.7) \quad \Sigma h(\infty) = 0, \quad h \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}.$$

By using (4.7) and the universal inequality

$$|x_1 + \dots + x_k|^2 \leq k(|x_1|^2 + \dots + |x_k|^2), \quad k \in \mathbb{N}$$

we get for arbitrary  $g, h \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$  the estimation

$$\begin{aligned} \|\Sigma g - \Sigma h\|_w^2 &= \int_{\mathbb{R}_+} \left| \sum_{i=1}^n h'_i(x) \Psi'_i \left( - \int_0^x h_i(\eta) d\eta \right) - \sum_{i=1}^n g'_i(x) \Psi'_i \left( - \int_0^x g_i(\eta) d\eta \right) \right. \\ &\quad \left. + \sum_{i=1}^n g_i(x)^2 \Psi''_i \left( - \int_0^x g_i(\eta) d\eta \right) - \sum_{i=1}^n h_i(x)^2 \Psi''_i \left( - \int_0^x h_i(\eta) d\eta \right) \right|^2 w(x) dx \\ &\leq 4n(I_1 + I_2 + I_3 + I_4), \end{aligned}$$

where we have put

$$\begin{aligned} I_1 &:= \sum_{i=1}^n \int_{\mathbb{R}_+} |h'_i(x)|^2 \left| \Psi'_i \left( - \int_0^x h_i(\eta) d\eta \right) - \Psi'_i \left( - \int_0^x g_i(\eta) d\eta \right) \right|^2 w(x) dx, \\ I_2 &:= \sum_{i=1}^n \int_{\mathbb{R}_+} \Psi'_i \left( - \int_0^x g_i(\eta) d\eta \right)^2 |h'_i(x) - g'_i(x)|^2 w(x) dx, \\ I_3 &:= \sum_{i=1}^n \int_{\mathbb{R}_+} g_i(x)^4 \left[ \Psi''_i \left( - \int_0^x g_i(\eta) d\eta \right) - \Psi''_i \left( - \int_0^x h_i(\eta) d\eta \right) \right]^2 w(x) dx, \\ I_4 &:= \sum_{i=1}^n \int_{\mathbb{R}_+} \Psi''_i \left( - \int_0^x h_i(\eta) d\eta \right)^2 (g_i(x)^2 - h_i(x)^2)^2 w(x) dx. \end{aligned}$$

Using (4.3) yields

$$I_1 \leq \sum_{i=1}^n L_i^2 \|h_i\|_w^2 \|g_i - h_i\|_{L^1(\mathbb{R}_+)}^2 \leq C_3^2 \sum_{i=1}^n L_i^2 \|h_i\|_w^2 \|g_i - h_i\|_w^2,$$

and  $I_2$  is estimated as

$$I_2 \leq \sum_{i=1}^n K_i^2 \|g_i - h_i\|_w^2.$$

Taking into account (4.3) and (4.4), we get

$$I_3 \leq \sum_{i=1}^n M_i^2 \|g_i^4 w\|_{L^1(\mathbb{R}_+)} \|g_i - h_i\|_{L^1(\mathbb{R}_+)}^2 \leq C_3^2 C_4 \sum_{i=1}^n M_i^2 \|g_i\|_w^4 \|g_i - h_i\|_w^2,$$

and by using Hölder's inequality and (4.4), we obtain

$$\begin{aligned} I_4 &\leq \sum_{i=1}^n L_i^2 \int_{\mathbb{R}_+} (g_i(x) + h_i(x))^2 w(x)^{\frac{1}{2}} (g_i(x) - h_i(x))^2 w(x)^{\frac{1}{2}} dx \\ &\leq \sum_{i=1}^n L_i^2 \| (g_i + h_i)^4 w \|_{L^1(\mathbb{R}_+)}^{\frac{1}{2}} \| (g_i - h_i)^4 w \|_{L^1(\mathbb{R}_+)}^{\frac{1}{2}} \\ &\leq 2C_4 \sum_{i=1}^n L_i^2 (\|g_i\|_w^2 + \|h_i\|_w^2) \|g_i - h_i\|_w^2, \end{aligned}$$

which gives us the desired estimation (4.6). For all  $h \in \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$  we have  $\Sigma h \in H_w^0$  by (4.6) and (4.7), and the map  $\Sigma : \Pi_{i=1}^n A_{H_w^0}^{\Psi_i} \rightarrow H_w^0$  is locally Lipschitz continuous by (4.6).  $\square$

By Proposition 4.5 we can, for given volatilities  $\sigma_i : \mathbb{R}_+ \times H_w \rightarrow H_w^0$  satisfying  $\sigma_i(\mathbb{R}_+ \times H_w) \subset A_{H_w^0}^{\Psi_i}$  for  $i = 1, \dots, n$ , define the drift term  $\alpha_{\text{HJM}}$  according to the HJM drift condition (2.4) by

$$(4.8) \quad \alpha_{\text{HJM}} := \Sigma \circ \sigma : \mathbb{R}_+ \times H_w \rightarrow H_w^0,$$

where  $\sigma = (\sigma_1, \dots, \sigma_n)$ .

Now, we are ready to establish the existence of Lévy term structure models on the space  $H_w$  of forward curves.

**4.6. Theorem.** *Let  $\sigma_i : \mathbb{R}_+ \times H_w \rightarrow H_w^0$  be continuous and satisfying  $\sigma_i(\mathbb{R}_+ \times H_w) \subset A_{H_w^0}^{\Psi_i}$  for  $i = 1, \dots, n$ . Assume there are  $M, L \geq 0$  such that for all  $i = 1, \dots, n$  and  $t \in \mathbb{R}_+$  we have*

$$\begin{aligned} \|\sigma_i(t, h)\|_w &\leq M, \quad h \in H_w \\ \|\sigma_i(t, h_1) - \sigma_i(t, h_2)\|_w &\leq L\|h_1 - h_2\|_w, \quad h_1, h_2 \in H_w. \end{aligned}$$

*Then, for each  $h_0 \in H_w$ , there exists a unique mild and a unique weak adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (1.5) with  $r_0 = h_0$  satisfying*

$$(4.9) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|r_t\|_w^2 \right] < \infty \quad \text{for all } T > 0.$$

*Proof.* By Proposition 4.5,  $\alpha_{\text{HJM}}$  maps into  $H_w^0$ , see (4.8). Since  $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{R}_+ \times H_w \rightarrow \Pi_{i=1}^n A_{H_w^0}^{\Psi_i}$  is continuous by assumption and  $\Sigma : \Pi_{i=1}^n A_{H_w^0}^{\Psi_i} \rightarrow H_w^0$  is continuous by Proposition 4.5, it follows that  $\alpha_{\text{HJM}} = \Sigma \circ \sigma$  is continuous. Moreover, by estimate (4.6), we obtain for all  $t \in \mathbb{R}_+$  and  $h_1, h_2 \in H_w$  the estimation

$$\begin{aligned} \|\alpha_{\text{HJM}}(t, h_1) - \alpha_{\text{HJM}}(t, h_2)\|_w &\leq C_5(1 + M)^2 \sum_{i=1}^n \|\sigma_i(t, h_1) - \sigma_i(t, h_2)\|_w \\ &\leq C_5(1 + M)^2 nL\|h_1 - h_2\|_w. \end{aligned}$$

Taking also into account Proposition 4.4, applying Theorem C.1 finishes the proof.  $\square$

As an immediate consequence, we get the existence of Lévy term structure models with constant direction volatilities.

**4.7. Corollary.** *Let  $\sigma_i : \mathbb{R}_+ \times H_w \rightarrow H_w^0$  be defined by  $\sigma_i(t, r) = \sigma_i(r) = \varphi_i(r)\lambda_i$ , where  $\lambda_i \in A_{H_w^0}^{\Psi_i}$  and  $\varphi_i : H_w \rightarrow [0, 1]$  for  $i = 1, \dots, n$ . Assume there is  $L \geq 0$  such that for all  $i = 1, \dots, n$  we have*

$$|\varphi_i(h_1) - \varphi_i(h_2)| \leq L\|h_1 - h_2\|_w, \quad h_1, h_2 \in H_w.$$

*Then, for each  $h_0 \in H_w$ , there exists a unique mild and a unique weak adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (1.5) with  $r_0 = h_0$  satisfying (4.9).*

*Proof.* For all  $h_1, h_2 \in H_w$  and all  $i = 1, \dots, n$  we get

$$\|\sigma_i(h_1) - \sigma_i(h_2)\|_w \leq L\|\lambda_i\|_w\|h_1 - h_2\|_w.$$

Also observing that  $\|\sigma_i(h)\|_w \leq \|\lambda_i\|_w$  for all  $h \in H_w$  and  $i = 1, \dots, n$ , the proof is a straightforward consequence of Theorem 4.6.  $\square$

The only assumption on the driving Lévy processes  $X^1, \dots, X^n$ , in order to apply the previous results, is the exponential moments condition (2.1). It is clearly satisfied for Brownian motions and Poisson processes.

There are also several purely discontinuous Lévy processes fulfilling (2.1), for instance generalized hyperbolic processes, which have been introduced by Barndorff-Nielsen [2], and their subclasses, namely the normal inverse Gaussian and hyperbolic processes. They have been applied to finance by Eberlein and co-authors in a series of papers, e.g. in [17].

Other purely discontinuous Lévy processes satisfying (2.1) are the generalized tempered stable processes, see [10, Sec. 4.5], which include Variance Gamma processes [43], CGMY processes [9] and bilateral Gamma processes [42].

Consequently, Theorem 4.6 applies to term structure models driven by any of the above types of Lévy processes.

## 5. CONCLUSION

We have established the existence of Lévy term structure models on two spaces of forward curves, namely in Section 3 on the Björk–Svensson space  $H_{\beta, \gamma}$ , on which  $\frac{d}{dx}$  is a bounded linear operator, and in Section 4 on the larger space  $H_w$ , where  $\frac{d}{dx}$  becomes unbounded.

In Section 3 it turned out that  $H_{\beta, \gamma}$  is too small to assert that  $\alpha_{\text{HJM}}$  given by the HJM drift-condition (2.4) lies in  $H_{\beta, \gamma}$ . However, for certain jump-diffusion models we have established existence and uniqueness on this space, see Proposition 3.6 and Proposition 3.9.

Our main results of Section 4 (Theorem 4.6 and Corollary 4.7), where we work on the larger space  $H_w$ , are applicable for a large range of driving Lévy processes, including mixtures of Brownian motion and Poisson processes, and purely discontinuous Lévy processes such as generalized hyperbolic processes and generalized tempered stable processes as well as several subclasses.

The existence results for Lévy term structure models are based on a general result for Hilbert space valued stochastic equations, see Theorem C.1 from the appendix. This result relies on two works of van Gaans [26, 27]. In order to make [27, Thm. 4.1] applicable for financial applications, where one is in particular interested in a solution with càdlàg trajectories, we have shown in the appendix that the stochastic integral constructed in van Gaans [27] has a càdlàg modification and we have analyzed when it coincides with the usual Itô-integral.

## APPENDIX A. OVERVIEW AND NOTATION

The goal of Appendix A – Appendix C is to provide an existence result for solutions of infinite dimensional stochastic differential equations, which is required in order to establish the existence of Lévy term structure models.

We intend to apply a result of van Gaans [27, Thm. 4.1]. However, as we shall see in Section B, the stochastic integral (G-)  $\int_0^t \Phi_s dX_s$  defined in van Gaans [27] is not consistent with the usual Itô-integral  $\int_0^t \Phi_s dX_s$ , which is used for financial modelling. This matters in view of applications to finance, because, as we have argued at the end of Section 1, we are in particular interested in a solution process with càdlàg paths.

In order to make [27, Thm. 4.1] applicable, we review the stochastic integral, which is defined in van Gaans [27], in Appendix B, show that it always possesses a càdlàg modification and analyze when it coincides with the usual Itô-integral. In Appendix C, we obtain the desired existence result concerning mild solutions, Theorem C.1, by applying [27, Thm. 4.1]. Using our findings of Appendix B, we

additionally show that the solution has a càdlàg modification and that it is also a weak solution.

Let  $H$  denote a separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle_H$  and associated norm  $\| \cdot \|_H$ . If there is no ambiguity, we shall simply write  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ .

Let  $T > 0$  be a finite time horizon. We denote by  $C_{\text{ad}}([0, T]; L^2(\Omega; H))$  the space of all continuous mappings  $\Phi : [0, T] \rightarrow L^2(\Omega; H)$  which are also adapted.

For two stochastic processes  $(\Phi_t)_{t \in [0, T]}$  and  $(\Psi_t)_{t \in [0, T]}$  we say that  $\Psi$  is a *modification* of  $\Phi$  if  $\mathbb{P}(\Phi_t = \Psi_t) = 1$  for all  $t \in [0, T]$ .

An adapted  $H$ -valued process  $(\Phi_t)_{t \in [0, T]}$  is called a *martingale* if

- $\mathbb{E}[\|\Phi_t\|] < \infty$  for all  $t \in [0, T]$ ;
- $\mathbb{E}[\Phi_t | \mathcal{F}_s] = \Phi_s$  ( $\mathbb{P}$ -a.s.) for all  $0 \leq s \leq t \leq T$ .

For the notion of conditional expectation of random variables having values in a separable Banach space, we refer to [13, Sec. 1.3].

An indispensable tool will be *Doob's martingale inequality*

$$(A.1) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right] \leq 4 \sup_{t \in [0, T]} \mathbb{E} [\|\Phi_t\|^2] = 4\mathbb{E} [\|\Phi_T\|^2],$$

valid for every  $H$ -valued càdlàg martingale  $\Phi$ , which is a consequence of Thm. 3.8 and Prop. 3.7 in [13].

## APPENDIX B. STOCHASTIC INTEGRATION

Let  $M$  be a real-valued Lévy martingale satisfying  $\mathbb{E}[M_1^2] < \infty$ . We recall how in this case the stochastic integral (G-)  $\int_0^t \Phi_s dM_s$ , in the sense of van Gaans [27, Sec. 3], is defined for  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ .

**B.1. Lemma.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . For each  $t \in [0, T]$ , there exists a unique random variable  $Y_t \in L^2(\Omega; H)$  such that for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$(B.1) \quad \mathbb{E} \left[ \left\| Y_t - \sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i}) \Phi_{t_i} \right\|^2 \right] < \varepsilon$$

for every partition  $0 = t_0 < t_1 < \dots < t_n = t$  with  $\sup_{i=0, \dots, n-1} |t_{i+1} - t_i| < \delta$ .

*Proof.* The assertion is a consequence of [27, Prop. 3.2.1].  $\square$

**B.2. Definition.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . Then the stochastic integral  $Y_t = (\text{G-}) \int_0^t \Phi_s dM_s$ ,  $t \in [0, T]$  is the stochastic process  $Y = (Y_t)_{t \in [0, T]}$  where every  $Y_t$  is the unique element from  $L^2(\Omega; H)$  such that (B.1) is valid.*

We observe that for every  $t \in [0, T]$  the stochastic integral (G-)  $\int_0^t \Phi_s dM_s$  is only determined up to a  $\mathbb{P}$ -null set. With regard to our applications to finance it arises the question if we can find a modification of the stochastic integral with càdlàg paths, a question which is not treated in [27].

Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . We define

$$(B.2) \quad I_t(\Phi) := (\text{G-}) \int_0^t \Phi_s dM_s, \quad t \in [0, T]$$

and the sequence of càdlàg adapted processes

$$(B.3) \quad I^n(\Phi) := \sum_{i=0}^{2^n-1} (M_{t_{i+1}^n} - M_{t_i^n}) \Phi_{t_i^n}, \quad n \in \mathbb{N}_0$$

where we set for  $n \in \mathbb{N}_0$  and  $i \in \{0, \dots, 2^n\}$

$$(B.4) \quad t_i^n := i2^{-n}T,$$

that is, we have a sequence of dyadic decompositions of the interval  $[0, T]$ . Note that each  $I^n(\Phi)$  is a martingale and that for each  $t \in [0, T]$  we have  $I_t^n(\Phi) \rightarrow I_t(\Phi)$  in  $L^2(\Omega; H)$  by Lemma B.1.

We let  $\mathcal{M}^2$  be the linear space of all càdlàg  $H$ -valued martingales  $(\Phi_t)_{t \in [0, T]}$ , which are square-integrable, i.e.  $\mathbb{E}[\|\Phi_t\|^2] < \infty$  for all  $t \in [0, T]$ , equipped with the norm

$$\|\Phi\|_2 = \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right]^{\frac{1}{2}}.$$

Note that by Doob's martingale inequality (A.1),  $\|\Phi\|_2$  is finite for every  $\Phi \in \mathcal{M}^2$ , and therefore  $\|\cdot\|_2$  defines a norm on the linear space  $\mathcal{M}^2$ . For the next result, we can almost literally follow the proof of [13, Prop. 3.9], which considers the continuous time case. For convenience of the reader, we provide the proof here.

**B.3. Proposition.** *The normed space  $(\mathcal{M}^2, \|\cdot\|_2)$  is a Banach space.*

*Proof.* Let  $(\Phi^n)$  be a Cauchy sequence in  $\mathcal{M}^2$ , i.e. for every  $\varepsilon > 0$  there is an index  $n_0 \in \mathbb{N}$  such that

$$(B.5) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t^n - \Phi_t^m\|^2 \right] < \varepsilon \quad \text{for all } n, m \geq n_0.$$

By the Markov inequality, there exists a subsequence  $(\Phi^{n_k})$  such that

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|\Phi_t^{n_{k+1}} - \Phi_t^{n_k}\| \geq 2^{-k} \right) \leq 2^{-k} \quad \text{for all } k \in \mathbb{N}.$$

The Borel-Cantelli lemma implies that for almost all  $\omega \in \Omega$  the sequence  $(\Phi^{n_k}(\omega))$  is a Cauchy sequence in the space of càdlàg functions on  $[0, T]$  equipped with the supremum-norm. Therefore,  $(\Phi^{n_k})$  converges  $\mathbb{P}$ -a.s. to an adapted process  $\Phi$ , uniformly on  $[0, T]$ . Hence,  $\Phi$  is càdlàg.

For each  $t \in [0, T]$ , the convergence  $\Phi_t^{n_k} \rightarrow \Phi_t$  is valid in  $L^2(\Omega; H)$ , because  $(\Phi_t^n)$  is a Cauchy sequence in  $L^2(\Omega; H)$  by (B.5). For  $0 \leq s \leq t \leq T$  and  $k \in \mathbb{N}$  we have  $\mathbb{E}[\Phi_t^{n_k} | \mathcal{F}_s] = \Phi_s^{n_k}$  ( $\mathbb{P}$ -a.s.), implying that  $\mathbb{E}[\Phi_t | \mathcal{F}_s] = \Phi_s$  ( $\mathbb{P}$ -a.s.). Consequently,  $\Phi$  is a martingale, and by Doob's martingale inequality (A.1), we get

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t - \Phi_t^n\|^2 \right] \leq 4 \sup_{t \in [0, T]} \mathbb{E} [\|\Phi_t - \Phi_t^n\|^2] = 4\mathbb{E} [\|\Phi_T - \Phi_T^n\|^2] \rightarrow 0$$

by (B.5) and completeness of  $L^2(\Omega; H)$ , i.e.  $\Phi^n \rightarrow \Phi$  in  $\mathcal{M}^2$ .  $\square$

In the following auxiliary result,  $\langle M, M \rangle$  denotes the *predictable quadratic co-variation* of the real-valued square-integrable martingale  $M$ , see [35, Thm. I.4.2].

**B.4. Lemma.** *Let  $0 = t_0 < \dots < t_n = T$  and  $Z_i : \Omega \rightarrow H$  be  $\mathcal{F}_{t_i}$ -measurable for  $i = 0, \dots, n-1$ . Then we have*

$$\mathbb{E} \left[ \left\| \sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i}) Z_i \right\|^2 \right] = \mathbb{E} \left[ \sum_{i=0}^{n-1} (\langle M, M \rangle_{t_{i+1}} - \langle M, M \rangle_{t_i}) \|Z_i\|^2 \right].$$

*Proof.* By using the identity  $\|x\|^2 = \langle x, x \rangle_H$ ,  $x \in H$  we obtain that

$$\begin{aligned} & \left\| \sum_{i=0}^{n-1} (M^{t_{i+1}} - M^{t_i}) Z_i \right\|^2 - \sum_{i=0}^{n-1} (\langle M, M \rangle^{t_{i+1}} - \langle M, M \rangle^{t_i}) \|Z_i\|^2 \\ &= 2 \sum_{\substack{i,j=0 \\ i < j}}^{n-1} (M^{t_{i+1}} - M^{t_i})(M^{t_{j+1}} - M^{t_j}) \langle Z_i, Z_j \rangle_H \\ & \quad + \sum_{i=0}^{n-1} [(M^{t_{i+1}})^2 - \langle M, M \rangle^{t_{i+1}} - (M^{t_i})^2 + \langle M, M \rangle^{t_i} - 2M_{t_i}(M^{t_{i+1}} - M^{t_i})] \|Z_i\|^2 \end{aligned}$$

is a martingale. Since  $\sum_{i=0}^{n-1} (\langle M, M \rangle^{t_{i+1}} - \langle M, M \rangle^{t_i}) \|Z_i\|^2$  is continuous and therefore predictable, the uniqueness of the predictable quadratic covariation yields

$$\left\langle \left\| \sum_{i=0}^{n-1} (M^{t_{i+1}} - M^{t_i}) Z_i \right\|, \left\| \sum_{i=0}^{n-1} (M^{t_{i+1}} - M^{t_i}) Z_i \right\| \right\rangle = \sum_{i=0}^{n-1} (\langle M, M \rangle^{t_{i+1}} - \langle M, M \rangle^{t_i}) \|Z_i\|^2,$$

proving the claimed equation.  $\square$

**B.5. Theorem.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . Then  $I(\Phi)$  has a modification which belongs to  $\mathcal{M}^2$  and, moreover,  $I^n(\Phi) \rightarrow I(\Phi)$  in  $\mathcal{M}^2$ .*

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Since  $\Phi : [0, T] \rightarrow L^2(\Omega; H)$  is uniformly continuous on the compact interval  $[0, T]$ , there exists  $\delta > 0$  such that

$$(B.6) \quad \mathbb{E} [\|\Phi_t - \Phi_s\|^2] < \frac{\varepsilon}{4T(c + \int_{\mathbb{R}} x^2 F(dx))}$$

for all  $s, t \in [0, T]$  with  $|t - s| < \delta$ , where  $c$  denotes the Gaussian part and  $F$  the Lévy measure of  $M$ . Choose  $n_0 \in \mathbb{N}$  such that  $2^{-n_0}T < \delta$ . For all  $n, m \in \mathbb{N}_0$  with  $n > m \geq n_0$  we obtain

$$I^n(\Phi) - I^m(\Phi) = \sum_{i=0}^{2^n-1} (M^{t_{i+1}^n} - M^{t_i^n})(\Phi_{t_i^n} - \Phi_{t_{j(i)}^n})$$

with  $j(i) \in \{0, \dots, i\}$  such that  $|t_i^n - t_{j(i)}^n| < 2^{-n_0}T < \delta$  for all  $i = 0, \dots, 2^n - 1$ . We obtain by Doob's martingale inequality (A.1), Lemma B.4 and (B.6) for all  $n, m \in \mathbb{N}_0$  with  $n > m \geq n_0$

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} \|I_t^n(\Phi) - I_t^m(\Phi)\|^2 \right] \leq 4\mathbb{E} \left[ \left\| \sum_{i=0}^{2^n-1} (M_{t_{i+1}^n} - M_{t_i^n})(\Phi_{t_i^n} - \Phi_{t_{j(i)}^n}) \right\|^2 \right] \\ &= 4 \sum_{i=0}^{2^n-1} \mathbb{E} \left[ (\langle M, M \rangle_{t_{i+1}^n} - \langle M, M \rangle_{t_i^n}) \|\Phi_{t_i^n} - \Phi_{t_{j(i)}^n}\|^2 \right] \\ &= 4 \left( c + \int_{\mathbb{R}} x^2 F(dx) \right) \sum_{i=0}^{2^n-1} (t_{i+1}^n - t_i^n) \mathbb{E} \left[ \|\Phi_{t_i^n} - \Phi_{t_{j(i)}^n}\|^2 \right] < \varepsilon. \end{aligned}$$

The latter identity is valid, because  $\langle M, M \rangle$  is the compensator of  $[M, M]$  by [35, Prop. I.4.50.b] and because the relation  $[M, M]_t = ct + \sum_{s \leq t} \Delta M_s^2$  is valid according to [35, Thm. I.4.52].

Thus, the sequence  $(I^n(\Phi))$  is a Cauchy sequence in  $\mathcal{M}^2$ . Proposition B.3 and Lemma B.1 complete the proof.  $\square$

For  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ , the integral with respect to  $dt$  can, according to [27, Lemma 3.6], be defined as a Riemann integral. More precisely:

**B.6. Lemma.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . For each  $t \in [0, T]$ , there exists a unique random variable  $Y_t \in L^2(\Omega; H)$  such that for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$(B.7) \quad \mathbb{E} \left[ \left\| Y_t - \sum_{i=0}^{n-1} (t_{i+1} - t_i) \Phi_{t_i} \right\|^2 \right] < \varepsilon$$

for every partition  $0 = t_0 < t_1 < \dots < t_n = t$  with  $\sup_{i=0, \dots, n-1} |t_{i+1} - t_i| < \delta$ .

*Proof.* Fix  $t \in [0, T]$  and let  $\varepsilon > 0$  be arbitrary. Since  $\Phi : [0, t] \rightarrow L^2(\Omega; H)$  is uniformly continuous on the compact interval  $[0, t]$ , there exists  $\delta > 0$  such that

$$(B.8) \quad \mathbb{E} [\|\Phi_s - \Phi_r\|^2] < \frac{\varepsilon}{t^2}$$

for all  $r, s \in [0, t]$  with  $|s - r| < \delta$ .

Let  $Z_1 = \{0 = t_0 < t_1 < \dots < t_n = t\}$  and  $Z_2 = \{0 = s_0 < s_1 < \dots < s_m = t\}$  be two decompositions satisfying  $\sup_{i=0, \dots, n-1} |t_{i+1} - t_i| < \delta$  and  $\sup_{i=0, \dots, m-1} |s_{i+1} - s_i| < \delta$ . Then there is a unique decomposition  $Z = \{0 = r_0 < r_1 < \dots < r_p = t\}$  such that  $Z = Z_1 \cup Z_2$ . Thus, we get

$$\sum_{i=0}^{n-1} (t_{i+1} - t_i) \Phi_{t_i} - \sum_{i=0}^{m-1} (s_{i+1} - s_i) \Phi_{s_i} = \sum_{i=0}^{p-1} (r_{i+1} - r_i) (\Phi_{a_i} - \Phi_{b_i})$$

with  $a_i \in Z_1$ ,  $b_i \in Z_2$  and  $|b_i - a_i| < \delta$  for all  $i = 0, \dots, p-1$ . We obtain by the Cauchy-Schwarz inequality and (B.8)

$$\begin{aligned} \mathbb{E} \left[ \left\| \sum_{i=0}^{n-1} (t_{i+1} - t_i) \Phi_{t_i} - \sum_{i=0}^{m-1} (s_{i+1} - s_i) \Phi_{s_i} \right\|^2 \right] &\leq \mathbb{E} \left[ \left( \sum_{i=0}^{p-1} (r_{i+1} - r_i) \|\Phi_{a_i} - \Phi_{b_i}\| \right)^2 \right] \\ &\leq \mathbb{E} \left[ \left( \sum_{i=0}^{p-1} (r_{i+1} - r_i) \right) \left( \sum_{i=0}^{p-1} (r_{i+1} - r_i) \|\Phi_{a_i} - \Phi_{b_i}\|^2 \right) \right] \\ &= t \sum_{i=0}^{p-1} (r_{i+1} - r_i) \mathbb{E} [\|\Phi_{a_i} - \Phi_{b_i}\|^2] < \varepsilon. \end{aligned}$$

By the completeness of  $L^2(\Omega; H)$ , the lemma is proven.  $\square$

**B.7. Definition.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . Then the integral  $Y_t = (G-) \int_0^t \Phi_s ds$ ,  $t \in [0, T]$  is the stochastic process  $Y = (Y_t)_{t \in [0, T]}$  where every  $Y_t$  is the unique element from  $L^2(\Omega; H)$  such that (B.7) is valid.*

Again, for every  $t \in [0, T]$  the integral  $(G-) \int_0^t \Phi_s ds$  is only determined up to a  $\mathbb{P}$ -null set. We shall prove the existence of a continuous modification.

Let  $\Phi \in C_{\text{ad}}([0, T], L^2(\Omega; H))$ . We define

$$(B.9) \quad J_t(\Phi) := (G-) \int_0^t \Phi_s ds, \quad t \in [0, T]$$

and the sequence of continuous adapted processes

$$(B.10) \quad J_t^n(\Phi) := \sum_{i=0}^{2^n-1} (t_{i+1}^n \wedge t - t_i^n \wedge t) \Phi_{t_i^n}, \quad n \in \mathbb{N}_0$$

where the  $t_i^n$  are defined in (B.4). Note that for each  $t \in [0, T]$  we have  $J_t^n(\Phi) \rightarrow J_t(\Phi)$  in  $L^2(\Omega; H)$  by Lemma B.6.



**B.8. Theorem.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$ . Then  $J(\Phi)$  has a continuous modification and, moreover,*

$$\sup_{t \in [0, T]} \|J_t^n(\Phi) - J_t(\Phi)\| \rightarrow 0 \quad \mathbb{P}\text{-a.s.}$$

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Since  $\Phi : [0, T] \rightarrow L^2(\Omega; H)$  is uniformly continuous on the compact interval  $[0, T]$ , there exists  $\delta > 0$  such that

$$(B.11) \quad \mathbb{E} [\|\Phi_t - \Phi_s\|^2] < \frac{\varepsilon}{T^2}$$

for all  $s, t \in [0, T]$  with  $|t - s| < \delta$ . Choose  $n_0 \in \mathbb{N}$  such that  $2^{-n_0}T < \delta$ . For all  $n, m \in \mathbb{N}_0$  with  $n > m \geq n_0$  we obtain

$$J^n(\Phi) - J^m(\Phi) = \sum_{i=0}^{2^n-1} (t_{i+1}^n \wedge t - t_i^n \wedge t) (\Phi_{t_i^n} - \Phi_{t_{j(i)}^n})$$

with  $j(i) \in \{0, \dots, i\}$  such that  $|t_i^n - t_{j(i)}^n| < 2^{-n_0}T < \delta$  for all  $i = 0, \dots, 2^n - 1$ . We obtain by the Cauchy-Schwarz inequality and (B.11) for all  $n, m \geq n_0$  with  $n > m$

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \|J_t^n(\Phi) - J_t^m(\Phi)\|^2 \right] &= \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \sum_{i=0}^{2^n-1} (t \wedge t_{i+1}^n - t \wedge t_i^n) (\Phi_{t_i^n} - \Phi_{t_{j(i)}^n}) \right\|^2 \right] \\ &\leq T \mathbb{E} \left[ \sup_{t \in [0, T]} \sum_{i=0}^{2^n-1} (t \wedge t_{i+1}^n - t \wedge t_i^n) \|\Phi_{t_i^n} - \Phi_{t_{j(i)}^n}\|^2 \right] \\ &= T \sum_{i=0}^{2^n-1} (t_{i+1} - t_i) \mathbb{E} [\|\Phi_{t_i^n} - \Phi_{t_{j(i)}^n}\|^2] < \varepsilon. \end{aligned}$$

By the Markov inequality, there exists a subsequence  $(J^{n_k}(\Phi))$  such that

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|J_t^{n_{k+1}}(\Phi) - J_t^{n_k}(\Phi)\| \geq 2^{-k} \right) \leq 2^{-k} \quad \text{for all } k \in \mathbb{N}.$$

The Borel-Cantelli lemma implies that for almost all  $\omega \in \Omega$  the sequence  $(J^{n_k}(\Phi)(\omega))$  is a Cauchy sequence in the space of continuous functions on  $[0, T]$  equipped with the supremum-norm. Therefore,  $(J^{n_k}(\Phi))$  converges  $\mathbb{P}$ -a.s. to an adapted process, uniformly on  $[0, T]$ , which is therefore continuous.

According to Lemma B.6, this limit process is a modification of the integral process  $J(\Phi)$ .  $\square$

Now let  $X$  be a real-valued Lévy process with  $\mathbb{E}[X_1^2] < \infty$ . Then it admits a unique decomposition  $X_t = M_t + bt$ , where  $M$  is a Lévy martingale satisfying  $\mathbb{E}[M_1^2] < \infty$  and  $b = \mathbb{E}[X_1]$ . According to [27, Def. 3.7], we set

$$(G-) \int_0^t \Phi_s dX_s := (G-) \int_0^t \Phi_s dM_s + b \cdot (G-) \int_0^t \Phi_s ds.$$

We shall also use the notation

$$G(\Phi)_t = (G-) \int_0^t \Phi_s dX_s, \quad t \in [0, T].$$

Note that  $G(\Phi) = I(\Phi) + b \cdot J(\Phi)$ , where  $I(\Phi)$  is defined in (B.2) and  $J(\Phi)$  is defined in (B.9). We also introduce  $G^n(\Phi) = I^n(\Phi) + b \cdot J^n(\Phi)$  for  $n \in \mathbb{N}_0$ , where  $I^n(\Phi)$  is defined in (B.3) and  $J^n(\Phi)$  is defined in (B.10).

For a predictable  $H$ -valued process  $\Phi$  and a real-valued semimartingale  $X$ , we can define the usual Itô-integral (developed e.g. in Jacod and Shiryaev [35] or Protter

[48])

$$\int_0^t \Phi_s dX_s,$$

which is used for financial modelling. The construction is just as for real-valued integrands, namely by defining the integral first for simple integrands and then extending it via the Itô-isometry. In order to get the Itô-isometry, it is vital that the state space  $H$  is a Hilbert space.

The construction of the stochastic integral in the more general situation, where the driving semimartingale may also be infinite dimensional, can be found in Métivier [44]. Da Prato and Zabczyk [13] and Carmona and Tehranchi [8] treat the case with infinite dimensional Brownian motion as integrator, in [8] also with a focus on interest rate models.

We also remark that the stochastic integral can still be defined on appropriate Banach spaces, so-called M-type 2 spaces. Then the integral is still a bounded linear operator, but no isometry, in general. We refer to [53] for further details.

We now observe that the integral (G-)  $\int_0^t \Phi_s dX_s$  of van Gaans [27] is not consistent with the usual stochastic integral  $\int_0^t \Phi_s dX_s$  used in financial modelling. As an example, let  $X$  be a standard Poisson process with values in  $\mathbb{R}$ . In Ex. 3.9 in [27] it is derived that

$$(G-) \int_0^t X_s dX_s = \frac{1}{2} (X_t^2 - X_t).$$

Apparently, this does not coincide with the pathwise Lebesgue-Stieltjes integral

$$\int_0^t X_s dX_s = \frac{1}{2} (X_t^2 + X_t),$$

but we have

$$(G-) \int_0^t X_s dX_s = \int_0^t X_{s-} dX_s,$$

showing that inconsistencies occur as soon as integrands with jumps are used. Indeed, we have the following general result about the relation between the integral of van Gaans and the usual Itô-integral:

**B.9. Theorem.** *Let  $\Phi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$  be left-continuous or càdlàg. Then we have for all  $t \in [0, T]$*

$$(B.12) \quad (G-) \int_0^t \Phi_s dX_s = \int_0^t \Phi_{s-} dX_s \quad \mathbb{P}\text{-a.s.}$$

*Proof.* If  $\Phi$  is left-continuous, we have  $\sup_{t \in [0, T]} \|G_t^n - G_t\| \rightarrow 0$  almost surely by Theorem B.5 and Theorem B.8, and therefore also in probability. For the usual Itô-integral we have  $\sup_{t \in [0, T]} \left\| G_t^n - \int_0^t \Phi_s dX_s \right\| \rightarrow 0$  in probability, which is proven as in the real-valued case, see e.g. [35, Prop. I.4.44]. Thus we obtain for all  $t \in [0, T]$

$$(B.13) \quad (G-) \int_0^t \Phi_s dX_s = \int_0^t \Phi_s dX_s \quad \mathbb{P}\text{-a.s.}$$

and, since  $\Phi$  is left-continuous, also relation (B.12).

If  $\Phi$  is càdlàg, we show that  $\Phi_-$  is a modification of  $\Phi$ , because then (B.12) is a consequence of (B.13) and Lemma B.11 below. Let  $t \in (0, T]$  be arbitrary and  $(t_n)$  be a sequence such that  $t_n \uparrow t$ . Since  $\Phi : [0, T] \rightarrow L^2(\Omega; H)$  is continuous, we deduce  $\mathbb{E}[\|\Phi_t - \Phi_{t_n}\|^2] \rightarrow 0$ . Thus there is a subsequence  $(n_k)$  with  $\|\Phi_t - \Phi_{t_{n_k}}\| \rightarrow 0$  almost surely, and therefore we have  $\mathbb{P}(\Phi_t = \Phi_{t-}) = 1$ .  $\square$

**B.10. Remark.** *If the driving process  $X$  is a (possibly infinite dimensional) Brownian motion, the equivalence of the van Gaans integral with the usual stochastic integral (see Da Prato and Zabczyk [13] for the infinite dimensional case) is provided in [26, Sec. 3].*

It remains to show the following auxiliary result, which we have used in the proof of Theorem B.9.

**B.11. Lemma.** *Let  $\Phi, \Psi \in C_{\text{ad}}([0, T]; L^2(\Omega; H))$  be such that  $\Psi$  is a modification of  $\Phi$ . Then  $G(\Psi)$  is a modification of  $G(\Phi)$ .*

*Proof.* Let  $t \in [0, T]$  be arbitrary. By hypothesis, we have  $\mathbb{P}(G_t^n(\Phi) = G_t^n(\Psi)) = 1$  for all  $n \in \mathbb{N}_0$ . Since  $G_t^n(\Phi) \rightarrow G_t(\Phi)$  and  $G_t^n(\Psi) \rightarrow G_t(\Psi)$  in  $L^2(\Omega; H)$  by Lemma B.1 and Lemma B.6, there is a subsequence  $(n_k)$  such that  $G_t^{n_k}(\Phi) \rightarrow G_t(\Phi)$  almost surely, and another subsequence  $n_{k_l}$  such that  $G_t^{n_{k_l}}(\Psi) \rightarrow G_t(\Psi)$  almost surely, showing that  $\mathbb{P}(G_t(\Phi) = G_t(\Psi)) = 1$ .  $\square$

### APPENDIX C. STOCHASTIC DIFFERENTIAL EQUATIONS

Now let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup in the separable Hilbert space  $H$ , i.e. a family of bounded linear operators  $S_t : H \rightarrow H$  such that

- $S_0 = \text{Id}$ ;
- $S_{s+t} = S_s S_t$  for all  $s, t \geq 0$ ;
- $\lim_{t \rightarrow 0} S_t h = h$  for all  $h \in H$ ;

with generator  $A : \mathcal{D}(A) \subset H \rightarrow H$ . By  $\|\cdot\|_{\mathcal{L}(H)}$  we denote the operator norm of a bounded linear operator. The semigroup  $(S_t)$  is called *contractive in  $H$*  if

$$\|S_t\|_{\mathcal{L}(H)} \leq 1, \quad t \geq 0$$

and *pseudo-contractive in  $H$*  if there is a constant  $\omega \geq 0$  such that

$$\|S_t\|_{\mathcal{L}(H)} \leq e^{\omega t}, \quad t \geq 0.$$

In this section, we intend to find mild solutions of stochastic differential equations of the type

$$(C.1) \quad \begin{cases} dr_t &= (Ar_t + \alpha(t, r_t))dt + \sum_{i=1}^n \sigma_i(t, r_{t-})dX_t^i, \\ r_0 &= h_0 \end{cases}$$

driven by real-valued Lévy processes  $X^1, \dots, X^n$  satisfying  $\mathbb{E}[(X_1^i)^2] < \infty$ ,  $i = 1, \dots, n$ , for each initial condition  $h_0 \in H$ , that is, a process  $(r_t)_{t \geq 0}$  satisfying

$$(C.2) \quad r_t = S_t h_0 + \int_0^t S_{t-s} \alpha(s, r_s) ds + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, r_{s-}) dX_s^i, \quad t \in \mathbb{R}_+.$$

We also intend to establish the existence of a weak solution  $(r_t)_{t \geq 0}$  to (C.1), i.e.  $(r_t)$  satisfies, for all  $\zeta \in \mathcal{D}(A^*)$ ,

$$(C.3) \quad \langle \zeta, r_t \rangle = \langle \zeta, h_0 \rangle + \int_0^t \left( \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(s, r_s) \rangle \right) ds + \sum_{i=1}^n \int_0^t \langle \zeta, \sigma_i(s, r_{s-}) \rangle dX_s^i$$

for each  $t \in \mathbb{R}_+$ . By convention, uniqueness of a solution to (C.1) is meant up to a modification. Here is our main existence and uniqueness result:

**C.1. Theorem.** *Let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup in  $H$ , and  $H_0 \subset H$  be a closed subspace such that  $(S_t)$  is pseudo-contractive in  $H_0$ . Let  $\alpha, \sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H_0$  be continuous. Assume there is constant  $L \geq 0$  such that*

$$(C.4) \quad \|\alpha(t, h_1) - \alpha(t, h_2)\| \leq L\|h_1 - h_2\|$$

$$(C.5) \quad \|\sigma_i(t, h_1) - \sigma_i(t, h_2)\| \leq L\|h_1 - h_2\|, \quad i = 1, \dots, n$$

for all  $t \in \mathbb{R}_+$  and all  $h_1, h_2 \in H$ . Then, for each  $h_0 \in H$ , there exists a unique mild and a unique weak adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (C.1) with  $r_0 = h_0$  satisfying

$$(C.6) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|r_t\|^2 \right] < \infty \quad \text{for all } T > 0.$$

*Proof.* Let  $h_0 \in H$  be arbitrary. We decompose each Lévy process  $X_t^i = M_t^i + b_i t$  into its martingale and finite variation part, where we notice that  $b_i = \mathbb{E}[X_1^i]$ . By [27, Thm. 4.1] there exists a unique adapted continuous function  $r : \mathbb{R}_+ \rightarrow L^2(\Omega; H)$  such that for all  $t \geq 0$

$$r_t = S_t h_0 + (\text{G-}) \int_0^t S_{t-s} \tilde{\alpha}(s, r_s) ds + \sum_{i=1}^n (\text{G-}) \int_0^t S_{t-s} \sigma_i(s, r_s) dM_s^i,$$

where  $\tilde{\alpha}(t, r) = \alpha(t, r) + \sum_{i=1}^n b_i \sigma_i(t, r)$ . By assumption,  $(S_t)$  is pseudo-contractive in  $H_0$ . Hence there exists a constant  $\omega \geq 0$  such that the  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  defined as

$$(C.7) \quad T_t := e^{-\omega t} S_t, \quad t \in \mathbb{R}_+$$

is contractive in  $H_0$ . By the Szekőfalvi-Nagy's theorem on unitary dilations (see e.g. [56, Thm. I.8.1], or [14, Sec. 7.2]), there exists another separable Hilbert space  $\mathcal{H}_0$  and a strongly continuous unitary group  $(U_t)_{t \in \mathbb{R}}$  in  $\mathcal{H}_0$  such that the diagram

$$\begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{U_t} & \mathcal{H}_0 \\ \uparrow \ell & & \downarrow \pi \\ H_0 & \xrightarrow{T_t} & H_0 \end{array}$$

commutes for every  $t \in \mathbb{R}_+$ , where  $\ell : H_0 \rightarrow \mathcal{H}_0$  is an isometric embedding (hence the adjoint operator  $\pi := \ell^*$  is the orthogonal projection from  $\mathcal{H}_0$  into  $H_0$ ), that is

$$(C.8) \quad \pi U_t \ell h = T_t h \quad \text{for all } t \in \mathbb{R}_+ \text{ and } h \in H_0.$$

Using (C.7), (C.8) and [27, Thm. 3.3.3] we obtain for all  $i = 1, \dots, n$  and  $t \geq 0$

$$\begin{aligned} (\text{G-}) \int_0^t S_{t-s} \sigma_i(s, r_s) dM_s^i &= (\text{G-}) \int_0^t e^{\omega(t-s)} T_{t-s} \sigma_i(s, r_s) dM_s^i \\ &= e^{\omega t} (\text{G-}) \int_0^t e^{-\omega s} \pi U_{t-s} \ell \sigma_i(s, r_s) dM_s^i = e^{\omega t} \pi U_t (\text{G-}) \int_0^t e^{-\omega s} U_{-s} \ell \sigma_i(s, r_s) dM_s^i. \end{aligned}$$

The integral process

$$(\text{G-}) \int_0^t e^{-\omega s} U_{-s} \ell \sigma_i(s, r_s) dM_s^i$$

has a càdlàg modification by Theorem B.5. Thus the process

$$(\text{G-}) \int_0^t S_{t-s} \sigma_i(s, r_s) dM_s^i$$

has a càdlàg modification, because  $(t, h) \mapsto U_t h$  is uniformly continuous on compact subsets, see e.g. [23, Lemma I.5.2].

A similar argumentation, using Theorem B.8, shows that

$$(G-) \int_0^t S_{t-s} \tilde{\alpha}(s, r_s) ds$$

has a continuous modification.

Therefore,  $(r_t)$  has a càdlàg modification, and, by Theorem B.9, it satisfies

$$r_t = S_t h_0 + \int_0^t S_{t-s} \tilde{\alpha}(s, r_s) ds + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, r_{s-}) dM_s^i, \quad t \geq 0.$$

Consequently,  $(r_t)_{t \geq 0}$  is a mild solution to (C.1), i.e. it satisfies (C.2). Introducing the processes

$$(C.9) \quad \Phi_t := \int_0^t S_{t-s} \tilde{\alpha}(s, r_s) ds,$$

$$(C.10) \quad \Psi_t^i := \int_0^t S_{t-s} \sigma_i(s, r_{s-}) dM_s^i, \quad i = 1, \dots, n$$

we have by our findings above

$$(C.11) \quad r_t = S_t h_0 + \Phi_t + \sum_{i=1}^n \Psi_t^i, \quad t \geq 0.$$

We fix an arbitrary  $T > 0$ . By (C.7), (C.8) and noting that  $\|\pi\|_{\mathcal{L}(\mathcal{H}_0; H_0)} = 1$  and  $\|U_t\|_{\mathcal{L}(\mathcal{H}_0)} \leq 1$  for all  $t \in [0, T]$ , we obtain for each  $i = 1, \dots, n$

$$(C.12) \quad \begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Psi_t^i\|^2 \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t S_{t-s} \sigma_i(s, r_{s-}) dM_s^i \right\|^2 \right] \\ & = \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| e^{\omega t} \pi U_t \int_0^t e^{-\omega s} U_{-s} \ell \sigma_i(s, r_{s-}) dM_s^i \right\|^2 \right] \\ & \leq e^{2\omega T} \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t e^{-\omega s} U_{-s} \ell \sigma_i(s, r_{s-}) dM_s^i \right\|^2 \right] < \infty. \end{aligned}$$

The latter expression is finite by Theorem B.5 and Theorem B.9.

We obtain by (C.7), (C.8), Hölder's inequality and Fubini's theorem (note that  $\|\tilde{\alpha}(t, r_t)\|^2$  is càdlàg and therefore  $\mathcal{B}[0, T] \otimes \mathcal{F}$ -measurable)

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t S_{t-s} \tilde{\alpha}(s, r_s) ds \right\|^2 \right] \\ & = \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| e^{\omega t} \pi U_t \int_0^t e^{-\omega s} U_{-s} \ell \tilde{\alpha}(s, r_s) ds \right\|^2 \right] \leq T e^{2\omega T} \mathbb{E} \left[ \int_0^T \|\tilde{\alpha}(t, r_t)\|^2 dt \right] \\ & \leq T e^{2\omega T} \int_0^T \mathbb{E} [\|\tilde{\alpha}(t, r_t)\|^2] dt \leq T^2 e^{2\omega T} \sup_{t \in [0, T]} \mathbb{E} [\|\tilde{\alpha}(t, r_t)\|^2] < \infty. \end{aligned}$$

The latter supremum is finite, because  $t \mapsto \mathbb{E} [\|\tilde{\alpha}(t, r_t)\|^2]$  is continuous on the compact interval  $[0, T]$ , as  $t \mapsto \tilde{\alpha}(t, r_t)$  is continuous by the continuity of  $r : [0, T] \rightarrow L^2(\Omega; H)$  and (C.4), (C.5). Since the solution process  $(r_t)$  is given by (C.11), we obtain, together with (C.12), that (C.6) is valid.

We proceed by showing that  $(r_t)_{t \geq 0}$  is also a weak solution to (C.1). Let  $\zeta \in \mathcal{D}(A^*)$  be arbitrary.

We define for arbitrary  $T \in \mathbb{R}_+$  the  $\mathcal{B}[0, T] \otimes \mathcal{P}$ -measurable functions  $H_i : [0, T] \times [0, T] \times \Omega \rightarrow \mathbb{R}$  as

$$H_i(a, t) := \begin{cases} \langle A^* \zeta, S_{a-t} \sigma_i(t, r_{t-}) \rangle, & a \geq t \\ 0, & a < t. \end{cases}$$

We obtain by the Cauchy-Schwarz inequality and the pseudo-contractivity of  $(S_t)$  in  $H_0$  that

$$(C.13) \quad |H_i(a, t)| \leq e^{\omega T} \|A^* \zeta\| \cdot \|\sigma_i(t, r_{t-})\|, \quad a, t \in [0, T].$$

The processes  $Y_t^i := (\int_0^T H_i^2(a, t) da)^{1/2}$  are left-continuous by (C.13) and Lebesgue's dominated convergence theorem, and therefore predictable. The Lévy martingales  $M^i$ , considered on  $[0, T]$ , belong to  $\mathcal{H}^2$  in the sense of the Definition in Protter [48, p. 156], because, by using [35, Thm. I.4.52],

$$\begin{aligned} \|M^i\|_{\mathcal{H}^2} &\leq \|M_i^c\|_{\mathcal{H}^2} + \|M_i^d\|_{\mathcal{H}^2} = \mathbb{E}[[M_i^c, M_i^c]_T]^{1/2} + \mathbb{E}[[M_i^d, M_i^d]_T]^{1/2} \\ &= (c_i T)^{1/2} + \mathbb{E}\left[\sum_{s \leq T} (\Delta M_s^i)^2\right]^{1/2} = (c_i T)^{1/2} + \left(T \int_{\mathbb{R}} x^2 F_i(dx)\right)^{1/2} < \infty, \end{aligned}$$

where we have decomposed  $M^i = M_i^c + M_i^d$  into its continuous and purely discontinuous martingale part, and where  $c_i$  denotes the Gaussian part and  $F_i$  the Lévy measure of  $M^i$ .

There is, by the assumed continuity of  $\sigma_1, \dots, \sigma_n$ , a constant  $C_T > 0$  such that  $\|\sigma_i(t, 0)\| \leq C_T$  for all  $t \in [0, T]$  and  $i = 1, \dots, n$ . Therefore, we get for all  $t \in [0, T]$ , all  $h \in H$  and all  $i = 1, \dots, n$  by (C.5)

$$(C.14) \quad \|\sigma_i(t, h)\| \leq \|\sigma_i(t, 0)\| + \|\sigma_i(t, h) - \sigma_i(t, 0)\| \leq (L \vee C_T)(1 + \|h\|).$$

By inequalities (C.13) and (C.14) we obtain

$$\begin{aligned} \mathbb{E}\left[\int_0^T (Y_t^i)^2 d[M^i, M^i]_t\right] &= \left(c_i + \int_{\mathbb{R}} x^2 F_i(dx)\right) \mathbb{E}\left[\int_0^T \int_0^T H_i^2(a, t) dadt\right] \\ &\leq T^2 (L \vee C_T)^2 e^{2\omega T} \|A^* \zeta\|^2 \left(c_i + \int_{\mathbb{R}} x^2 F_i(dx)\right) \mathbb{E}\left[\sup_{t \in [0, T]} (1 + \|r_t\|)^2\right]. \end{aligned}$$

Thus, the processes  $Y^i$  are  $(\mathcal{H}^2, M^i)$  integrable in the sense of the Definition in Protter [48, p. 165], because from Hölder's inequality and (C.6) we infer

$$\mathbb{E}\left[\sup_{t \in [0, T]} (1 + \|r_t\|)^2\right] \leq 1 + 2\mathbb{E}\left[\sup_{t \in [0, T]} \|r_t\|^2\right]^{1/2} + \mathbb{E}\left[\sup_{t \in [0, T]} \|r_t\|^2\right] < \infty.$$

Consequently, we have  $Y^i \in L(M^i)$ , that is each  $Y^i$  is  $M^i$  integrable in the sense of Protter [48, p. 165], and therefore we may apply the Fubini Theorem, see Thm. IV.65 in [48], for the integrands  $H_i$ . Using the Fubini Theorem and [58, Lemma VII.4.5(a)], we obtain for each  $i = 1, \dots, n$

$$\begin{aligned} \int_0^t \langle A^* \zeta, \Psi_s^i \rangle ds &= \int_0^t \int_0^s \langle A^* \zeta, S_{s-u} \sigma_i(u, r_{u-}) \rangle dM_u^i ds \\ &= \int_0^t \left\langle A^* \zeta, \int_u^t S_{s-u} \sigma_i(u, r_{u-}) ds \right\rangle dM_u^i = \int_0^t \left\langle \zeta, A \int_0^{t-u} S_s \sigma_i(u, r_{u-}) ds \right\rangle dM_u^i \\ &= \int_0^t \langle \zeta, S_{t-u} \sigma_i(u, r_{u-}) - \sigma_i(u, r_{u-}) \rangle dM_u^i = \langle \zeta, \Psi_t^i \rangle - \int_0^t \langle \zeta, \sigma_i(s, r_{s-}) \rangle dM_s^i, \end{aligned}$$

where the  $\Psi^i$  are defined in (C.10). An analogous calculation, using the standard Fubini theorem, gives us

$$\int_0^t \langle A^* \zeta, \Phi_s \rangle ds = \langle \zeta, \Phi_t \rangle - \int_0^t \langle \zeta, \tilde{\alpha}(s, r_s) \rangle ds,$$

where  $\Phi$  is defined in (C.9), and finally, we get, by taking into account [58, Lemma VII.4.5(a)] again,

$$\int_0^t \langle A^* \zeta, S_s h_0 \rangle ds = \left\langle \zeta, A \int_0^t S_s h_0 ds \right\rangle = \langle \zeta, S_t h_0 \rangle - \langle \zeta, h_0 \rangle.$$

Together with (C.11), the latter three identities show that

$$\langle \zeta, r_t \rangle = \langle \zeta, h_0 \rangle + \int_0^t \left( \langle A^* \zeta, r_s \rangle + \langle \zeta, \tilde{\alpha}(s, r_s) \rangle \right) ds + \sum_{i=1}^n \int_0^t \langle \zeta, \sigma_i(s, r_{s-}) \rangle dM_s^i$$

for all  $t \in [0, T]$ . Since  $T \in \mathbb{R}_+$  was arbitrary,  $(r_t)_{t \geq 0}$  is a weak solution to (C.1), as it fulfills (C.3).

It remains to show that this weak solution is unique. Let  $(r_t)_{t \geq 0}$  be any adapted càdlàg weak solution to (C.1), i.e.  $(r_t)$  satisfies (C.3) for all  $\zeta \in \mathcal{D}(A^*)$ . Let  $\zeta \in \mathcal{D}(A^*)$  and  $g \in C^1([0, T]; \mathbb{R})$  for an arbitrary  $T \in \mathbb{R}_+$ . By the definition of the quadratic co-variation  $[X, Y]$ , see e.g. [35, Def. I.4.45], we obtain

$$\langle g(t) \zeta, r_t \rangle = \langle g(0) \zeta, h_0 \rangle + \int_0^t g(s) d\langle \zeta, r_s \rangle + \int_0^t \langle \zeta, r_s \rangle dg(s) + [g, \langle \zeta, r \rangle]_t.$$

Since  $g \in C^1([0, T]; \mathbb{R})$ , we have  $[g, \langle \zeta, r \rangle] = 0$  according to [35, Prop. 4.49.d]. Therefore and because of (C.3), we get

$$\begin{aligned} \langle g(t) \zeta, r_t \rangle &= \langle g(0) \zeta, h_0 \rangle + \int_0^t \left( \langle g'(s) \zeta + A^* g(s) \zeta, r_s \rangle + \langle g(s) \zeta, \alpha(s, r_s) \rangle \right) ds \\ &\quad + \sum_{i=1}^n \int_0^t \langle g(s) \zeta, \sigma_i(s, r_{s-}) \rangle dX_s^i. \end{aligned}$$

Since the set  $\{t \mapsto g(t) \zeta \mid g \in C^1([0, T]; \mathbb{R})\}$  is dense in  $C^1([0, T]; \mathcal{D}(A^*))$ , we deduce

$$\begin{aligned} \langle g(t), r_t \rangle &= \langle g(0), h_0 \rangle + \int_0^t \left( \langle g'(s) + A^* g(s), r_s \rangle + \langle g(s), \alpha(s, r_s) \rangle \right) ds \\ &\quad + \sum_{i=1}^n \int_0^t \langle g(s), \sigma_i(s, r_{s-}) \rangle dX_s^i \end{aligned}$$

for all  $g \in C^1([0, T]; \mathcal{D}(A^*))$ , where we recall that  $T \in \mathbb{R}_+$  was arbitrary. Defining  $g \in C^1([0, t]; \mathcal{D}(A^*))$  for an arbitrary  $t \in \mathbb{R}_+$  and an arbitrary  $\zeta \in \mathcal{D}(A^*)$  as  $g(s) := S_{t-s}^* \zeta$ ,  $s \in [0, t]$ , we obtain  $g'(s) = -A^* g(s)$ , and hence

$$\langle \zeta, r_t \rangle = \langle \zeta, S_t h_0 \rangle + \int_0^t \langle \zeta, S_{t-s} \alpha(s, r_s) \rangle ds + \sum_{i=1}^n \int_0^t \langle \zeta, S_{t-s} \sigma_i(s, r_{s-}) \rangle dX_s^i.$$

Since  $\mathcal{D}(A^*)$  is dense in  $H$ , the process  $(r_t)_{t \geq 0}$  is also a mild solution to (C.1), i.e. it satisfies (C.2), proving the desired uniqueness.  $\square$

In the special situation where  $A \in \mathcal{L}(H)$ , i.e.  $A$  is a bounded linear operator, we can now easily establish the existence of a strong solution  $(r_t)_{t \geq 0}$  to (C.1), that is we have

$$(C.15) \quad r_t = h_0 + \int_0^t \left( A r_s + \alpha(s, r_s) \right) ds + \sum_{i=1}^n \int_0^t \sigma_i(s, r_{s-}) dX_s^i, \quad t \geq 0.$$

**C.2. Corollary.** *Let  $A \in \mathcal{L}(H)$  be a bounded linear operator and let  $\alpha, \sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H$  be continuous. Assume there is constant  $L \geq 0$  such that (C.4) and (C.5) are satisfied for all  $t \in \mathbb{R}_+$  and  $h_1, h_2 \in H$ . Then, for each  $h_0 \in H$ , there exists a unique strong adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (C.1) with  $r_0 = h_0$  satisfying (C.6).*

*Proof.* The operator  $A$  is generated by the semigroup  $S_t = e^{tA}$ , which is pseudo-contractive, because

$$\|S_t\|_{\mathcal{L}(H)} \leq e^{t\|A\|_{\mathcal{L}(H)}}, \quad t \geq 0.$$

By Theorem C.1, for each  $h_0 \in H$ , there exists a unique weak adapted càdlàg solution  $(r_t)_{t \geq 0}$  to (C.1) with  $r_0 = h_0$  satisfying (C.6), which also fulfills (C.15) by the boundedness of  $A$ , showing that  $(r_t)$  is a strong solution to (C.1).  $\square$

We close this section with a couple of remarks. Actually, [27, Thm. 4.1] is not explicitly proven in [27]. We quote [27, p. 19]: "For a proof of Theorem 4.1 one can follow almost literally the proofs of Theorem 4.1 and Theorem 4.2 in [26], ...". The mentioned result, [26, Thm. 4.1], is an analogous result for stochastic equations driven by an infinite dimensional Brownian motion.

Note that the existence result of van Gaans [27, Thm. 4.1] demands no further assumptions on the  $C_0$ -semigroup. In contrast, we require the pseudo-contractivity of  $(S_t)$  in a closed subspace in order to prove that the solution possesses a càdlàg modification.

The idea to use the Szekőfalvi-Nagy's theorem on unitary dilations in order to overcome the difficulties arising from stochastic convolutions, is due to Hausenblas and Seidler, see [31] and [30].

Without using the Szekőfalvi-Nagy's theorem, Baudoin and Teichmann [3] consider stochastic equations on separable Hilbert spaces equipped with a strongly continuous group, in Sec. 3 of their article also with focus on interest rate theory.

For every pseudo-contractive semigroup  $(S_t)$ , stochastic convolutions  $\int_0^t S_{t-s} \Phi_s dM_s$  with respect to a square-integrable, càdlàg martingale  $M$  have a càdlàg modification, which is due to Kotelenetz [41]. We use the Szekőfalvi-Nagy's theorem on unitary dilations in order to get a càdlàg modification, because we deal with the stochastic integral (G-)  $\int_0^t S_{t-s} \Phi_s dM_s$  defined in van Gaans [27, Sec. 3].

Recently, there has been growing interest in stochastic differential equations of the type (C.1) with jump noise terms. As a result, a few related papers [1, 39, 40, 28, 29, 45, 50] and the forthcoming textbook [49] have been written, but mostly with other fields of applications than finance.

During the revision of this paper we became aware of the recent preprint [50], where the authors derived independently similar results. But they work on different function spaces where the forward curve is not necessarily continuous and thus the short rate is not well defined. Moreover, they only consider volatilities of composition type, that is  $\sigma_i(t, r)(x) = g_i(t, x, r(x))$  with deterministic functions  $g_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ .

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