Sensitivity of life insurance reserves via Markov semigroups

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Motivation

Why sensitivities?

- Risk management means managing the available capital (CFO and CRO responsibility)
- Regulatory capital requirements (e.g., standard model in Solvency II) are based on parameter scenarios
- Thus, sensitivities of insurance reserves with respect to the valuation basis are of particular interest
- Understanding the sensitivities is key to premium calculation, reserving and ultimately the survival of the insurer
The reserve is usually defined as the discounted expected benefit payments less than the discounted expected premium payments (equivalence principle).

In 1875 Thiele devised a differential equation (and difference equation) for the evolution of the reserve

$$\frac{d}{dt} V_t = \pi_t - b_t\mu_{x+t} + (r + \mu_{x+t})V_t.$$ 

Unification of the time discrete and continuous case in a stochastic integral equation [MS97]
Thiele’s differential equation II

- Combination with developments in financial mathematics (Black-Scholes, term-structure models, ...) leads to generalized Thiele equations, cf. [Nor91] or [Ste06]
- These generalizations model modern life insurance products whose benefits explicitly depend on capital markets
- In product design and capital requirements current attention shifts towards worst-case analyses with respect to the valuation basis
- Examples include [Chr11b], [Chr11a], [Chr10] and [CS11]
Our model life insurance contract

- We consider a multi-state life insurance policy with distribution of a surplus as in [Ste07]
- The surplus can be invested in a risk-free asset and a risky asset, the latter being modelled by an Itô process
- The reserve satisfies a system of partial differential equations (PDEs)
- Objective: solve the PDEs by semigroup techniques and then assess sensitivities
The aim is to investigate the Thiele PDE using linear operators

- Basic idea is to express economic forces by linear operators
- Motivation: quantum mechanics and operator algebras
- Key results
  - Uniform continuity of the reserve with respect to financial, mortality and payment assumptions
  - Pointwise bounds on the gradient of the reserve as a function of the surplus
  - Factorization of the reserve into risk types (financial, insurance, payment)
- Basis for treatment of polynomial processes (including Lévy and affine processes)
Selected other approaches to sensitivities

Key ideas from the literature

- Valuation basis depends on a single parameter $\theta$. Differentiate Thiele’s equation with respect to $\theta$ and solve ensuing PDE. Cf. [KN03]

- Valuation basis lives in a Hilbert space. Consider the reserve as a functional of the valuation basis and apply Fréchet derivative with respect to valuation basis. Cf. [Chr08]
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Start with the reserve as a conditional expectation

Consider a life insurance policy with benefit payments that depend on a surplus. The surplus can be invested in a risk-free and a risky asset. All processes are (jointly) Markov:

- $Z_t$: process with values in $\{1, \ldots, n\}$, state of the insured person
- $X_t$: process for the value of the surplus (SDE)
- $B_t$: process for benefit payments
- $D_t$: process for dividend payments from the surplus

Define the *market reserve* $V^j$ of the contract in state $j$ as

$$
V^j(t, x) = \mathbb{E} \left[ \int_t^T e^{(s-t)r} d(B + D)(s) \bigg| Z(t) = j, X(t) = x \right],
$$

with the policy terminating at time $T$
The reserve satisfies a PDE system 1

The reserve vector \( \mathbf{V} = (V^1, \ldots, V^n)^\top \) satisfies

\[
\begin{align*}
0 &= \partial_t V^j(t, x) + D^j(t) V^j(t, x) + \beta^j(t, x) - rV^j(t, x), \\
0 &= V^j(T, x),
\end{align*}
\]

on \([0, T] \times \mathbb{R}\) where

\[
D^j(t) = \frac{1}{2} \pi(t, x)^2 \sigma^2 x^2 \partial_x^2 + (r x + c^j(t) - \delta^j(t, x)) \partial_x
\]
\[
+ \sum_{k \neq j} \mu^{jk}(t) (V^k(t, x + c^{jk}(t) - \delta^{jk}(t, x)) - V^j(t, x)),
\]

\[
\beta^j(t, x) = b^j(t) + \delta^j(t, x) + \sum_{k \neq j} \mu^{jk}(t) (b^{jk}(t) + \delta^{jk}(t, x)).
\]

For the derivation of these equations see [Ste06, Ste07]
The Thiele PDE

The reserve satisfies a PDE system II

Meaning of the variables and coefficients

\[ t = \text{time} \]
\[ x = \text{value of the surplus} \]
\[ T = \text{maturity of the contract} \]
\[ V^j(t, x) = \text{reserve in state } j \]
\[ r = \text{constant risk-free interest rate} \]
\[ \pi(t, x) = \text{surplus share invested in the risky asset} \]
\[ \sigma = \text{diffusion coefficient for the risky asset} \]
\[ b^{jk}(t), b^j(t) = \text{benefit payments} \]
\[ \mu^{jk}(t) = \text{transition intensities} \]
\[ \delta^j(t, x), \delta^{jk}(t, x) = \text{dividends from the surplus} \]
\[ c^j(t), c^{jk}(t) = \text{contributions to the surplus} \]
The Thiele PDE

The reserve satisfies a PDE system III

Hypothesis

(i) Coefficients of the differential operators
(a) there is a $\pi_0 > 0$ with $\pi(x) \geq \pi_0$ for all $x \in \mathbb{R}_+$,
(b) $\pi, c^j, \delta^j$ are in $C^{\alpha/2,\alpha}_{loc}([0, T] \times \mathbb{R}_+)$ for a $\alpha \in (0, 1)$,
(c) the function $\pi$ is bounded and $c^j \geq 0$.

(ii) Regularity of payments, dividends and intensities
(a) $b^j$ and $\mu^{jk}$ belong to $C([0, T])$,
(b) $\delta^{jk}$ belongs to $C^{0,\alpha}(\mathbb{R}_+) \times \mathbb{R}_+)$ for all $j, k$.

(iii) Boundedness of dividend payments
(a) there is a constant $k > 0$ with $0 \leq \delta^j(t, x) \leq kx$,
(b) the term $\delta^j(t, x) \frac{-\log x}{x(1+log^2 x)}$ is bounded for $x \to 0$,
(c) for all $x, t$ we have $x + c^{jk}(t) - \delta^{jk}(t, x) \geq 0$. 
Relaxation of assumptions I

Coefficients of the differential operators

(a) there is a $\pi_0 > 0$ with $\pi(x) \geq \pi_0$ for all $x \in \mathbb{R}_+$,
(b) $\pi, c^j, \delta^j$ are in $C^{\alpha/2, \alpha}_{loc}([0, T] \times \mathbb{R}_+)$ for a $\alpha \in (0, 1)$,
(c) the function $\pi$ is bounded and $c^j \geq 0$.

- Allow for $\pi(x) \geq 0$ i.e., surrender uniform ellipticity, the PDE is degenerate
- The analysis is done by regularizing the equation i.e., one considers $\mathcal{A}^j + \epsilon \Delta$ for $\epsilon > 0$ and the Laplace operator $\Delta$
- This leads to a solution $V_{\epsilon}$. Now consider $\epsilon \to 0$ and show weak convergence e.g., in $L^2$
Relaxation of assumptions II

Regularity of payments, dividends and intensities

(a) $b^j$ and $\mu^{jk}$ belong to $C([0, T])$,
(b) $\delta^{jk}$ belongs to $C^{0,\alpha}([0, T] \times \mathbb{R})$ for all $j, k$.

- Allow for measurable coefficients
- Leads to solution in $L^\infty$ or $L^p$-spaces
Relaxation of assumptions III

Boundedness of dividend payments

(a) there is a constant \( k > 0 \) with \( 0 \leq \delta^j(t, x) \leq kx \),

(b) the term \( \delta^j(t, x) \frac{-\log x}{x(1 + \log^2 x)} \) is bounded for \( x \to 0 \),

(c) for all \( x, t \) we have \( x + c^{jk}(t) - \delta^{jk}(t, x) \geq 0 \).

- Crucial in our framework
- However, no practical limitations as the important case of \( \delta^* \) linear in \( x \) is covered
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Solution semigroups of PDEs I

The basic example is the heat equation on \( \mathbb{R}^n \):

\[
\partial_t u(t, x) = \Delta u(t, x), \\
u(0, x) = g(x).
\]

View this as an abstract evolution equation in \( C^{1,2}(\mathbb{R}^n) \) by regarding \( u(t, \cdot) \) as and element of \( C([0, T]; C^2(\mathbb{R}^n)) \). Then rewrite the PDE as

\[
\partial_t u(t) = \Delta u(t), \\
u(0) = g,
\]

a first-order ordinary differential equation in \( t \). Formally solve this as

\[
u(t) = e^{t\Delta} u(0).\]

Does this make sense?
Properties of semigroups

Operator families of the type \( e^{tA} \) acting on a Banach space \( X \), should have the following properties:

(i) \( e^{tA} e^{sA} = e^{(t+s)A} \) for \( t, s > 0 \) (semigroup property)

(ii) \( \lim_{t \to 0} e^{tA}x = x \) for all \( x \in X \) (strong continuity)

(iii) \( \partial_t a^{tA} = Ae^{tA} = e^{tA}A \) (solution of the PDE)

In our case we will not have strong continuity, still \( e^{0A} = id \).

See also [Ama95], [Paz83], [EN00], [Lun95] for the construction of semigroups and their application to PDEs.
The generator of a semigroup

Let \( T(t) \) be a strongly continuous semigroup acting on a Banach space \( X \). Define

\[
D(A) = \left\{ f \in X : \frac{T(t)f - f}{t} \text{ converges in norm for } t \to 0_+ \right\}
\]

and set

\[
A(f) = \lim_{t \to 0_+} \frac{T(t)f - f}{t} \text{ for } f \in D(A).
\]

We call \( A \) the \textit{(infinitesimal) generator} of the semigroup and \( D(A) \) the \textit{domain} of \( A \). \( D(A) \) is a linear subspace of \( X \) and \( A \) is a linear map \( D(A) \to X \). Usually, \( D(A) \) is very hard to identify precisely.
Construction of semigroups

There are several ways to construct operators $e^{tA}$:  
- as the solution to $\partial_t u = Au$  
- as a Taylor series in case $A : X \to X$ is bounded  
\[ e^{tA} = I + tA + \frac{1}{2} t^2 A^2 + \cdots \]
- by functional calculus on a Banach algebra  
- by a Cauchy integral, if the resolvent $(\lambda - A)^{-1}$ is bounded  
\[ e^{tA} = \frac{i}{2\pi} \oint_{\Gamma} e^{\lambda t} (\lambda - A)^{-1} d\lambda. \]
- by the theorems of Hille-Yosida, Ray, Phillipps, etc.
Relationship with stochastic processes

Morally: let \((X_t)_{t \geq 0}\) be a stochastic process with state space \(\mathbb{R}^n\). Fix \(x \in \mathbb{R}^n\). The semigroup \(T(t)\) for \(X_t\) acts on functions \(u: \mathbb{R}^n \to \mathbb{R}\) as follows

\[
[T(t)u](x) = \mathbb{E}^x(u(X_t)).
\]

The generator \(A\) is given by

\[
Au = \lim_{t \to 0} \frac{T(t)u - u}{t}.
\]

In case of a Brownian motion with drift \(A\) has the form

\[
A = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i}.
\]
Generalization to evolution families

So far we had time-independent (autonomous) generators. Now generalization to time-dependence which leads to evolution families. A family of linear operators \( \{ G(t, s) : 0 \leq s \leq t \leq T \} \) in \( \mathcal{B}(X) \) is called evolution family if

(i) \( G(t, s)G(s, r) = G(t, r) \) for \( 0 \leq r \leq s \leq t \leq T \) and \( G(s, s) = id \)

(ii) \( G(t, s) \) maps \( X \) to \( D \) with \( D \) the domain of \( A(t) \), where we assume that all \( A(t) \) have the same domain

(iii) The map \( t \mapsto G(t, s) \) is differentiable on \( (s, T] \) with values in \( \mathcal{B}(X) \) and for \( 0 \leq s \leq t \leq T \) we have \( \partial_t G(t, s) = A(t)G(t, s) = G(t, s)A(t) \).
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Consider Thiele as an abstract evolution equation

Step 1: define a time-dependent linear operator \( T = T(t) \) acting on \( C_b([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n \) by

\[
T = \begin{pmatrix}
- \sum_{k \neq 1} \mu^{1k} 1 & \mu^{12} T^{12} & \mu^{13} T^{13} & \ldots & \mu^{1n} T^{1n} \\
\mu^{21} T^{21} & - \sum_{k \neq 2} \mu^{2k} 1 & \mu^{23} T^{23} & \ldots & \mu^{2n} T^{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu^{n1} T^{n2} & \mu^{n2} T^{n2} & \mu^{n3} T^{n3} & \ldots & - \sum_{k \neq n} \mu^{nk} 1
\end{pmatrix}
\]

Here, \( \mu^{jk} = \mu^{jk}(t) \) and the \( T^{jk} = T^{jk}(t) \) are linear operators:

\[
(T^{ik} f)(t, x) = f(t, x + c^{jk}(t) - \delta^{jk}(t, x))
\]

Morally: insurance risk expressed by \( T \)
Consider Thiele as an abstract evolution equation II

Step 2: spacetime transformation $\tau = T - t$ and $y = \log x$. Define

$$A^j = \frac{1}{2} \pi^2 \sigma^2 \partial_y^2 + (r + (c^j - \delta^j) e^{-y} - \frac{1}{2} \pi \sigma^2) \partial_y.$$  

With the diagonal operator $A = \begin{pmatrix} A^1 \\ \vdots \\ A^n \end{pmatrix}$ the reserve vector satisfies

$$\begin{align*}
\partial_\tau V &= A(\tau)V + TV - rV + e^{r\tau} \beta \\
V(0) &= 0,
\end{align*}$$

an abstract initial value problem on a suitable Banach space.
Formulation as an integral equation with semigroups

Let $G$ be the evolution family generated by $A$ i.e., a family of linear operators $G(\tau, s)$ on a suitable space such that

$$G(\tau, s)G(s, \rho) = G(\tau, \rho)$$

for $\rho \leq s \leq \tau$ and $G(\tau, \tau) = id$

The existence of $G$ is non-trivial as $A$ has exponentially growing first-order coefficients

$V$ is a *mild solution* of (1) if the Duhamel formula is satisfied

$$V(\tau) = \int_0^\tau G(\tau, s) \left[ T(s)V(s) + e^{-r(\tau-s)}\beta(s) \right] \, ds \quad (2)$$
The PDE has a unique mild solution I

**Theorem**

Assume the coefficients of $\mathcal{A}$ are in $C^{\alpha/2,1+\alpha}$ for $\alpha \in (0, 1)$. Then there is a unique mild solution $V$ in the space $C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$.

The Duhamel decomposition (2) of $V$ shows the factorization of the integrand into operators:

1. market risks from the investment in the risky asset as represented by $G$,
2. the effect of net payments represented by the multiplication operator $\beta$, and
3. insurance risk represented by $T$
The PDE has a unique mild solution II

Express the solution explicitly in terms of a Neumann series (Dyson series in physics, Peano series in matrix analysis). Let

\[ f(\tau) = \int_0^\tau e^{r\beta(s)} \, ds, \]

then

\[ V = e^{-r\tau} \left( f + GT\#f + GT\#GT\#f + \cdots \right) \quad (3) \]

under the operation

\[ (GT\#\xi)(\tau) = \int_0^\tau G(\tau, s)T(s)\xi(s) \, ds. \]

The series converges in \( C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n \). This leads to a conceptual explanation how the reserve depends on payments. The series is also an asymptotic expansion in \( \tau \) and can be used to approximate \( V \).
Main results

Continuous dependence of the reserve on the data

**Theorem**

Let \( Y_1 = C_b(\mathbb{R}) \otimes \mathbb{R}^n \), \( Y_2 = C_b([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n \), \( \varphi(\tau) = \int_0^\tau e^{-r(\tau-s)}ds \), and \( \hat{T} = \sup_{\tau} ||T(\tau)|| \). Then

(i) **growth**: \( ||V(\tau)||_{Y_1} \leq ||\beta||_{Y_2} \left( \hat{T} e^{\hat{T}\tau} \int_0^\tau \varphi(s)ds + \varphi(\tau) \right) \)

(ii) **dependence on payments**: \( ||V_1(\tau) - V_2(\tau)||_{Y_1} \leq ||\beta_1 - \beta_2||_{Y_2} \left( \hat{T} e^{\hat{T}\tau} \int_0^\tau \varphi(s)ds + \varphi(\tau) \right) \)

(iii) **dependence on insurance risk**: \( ||V_1(\tau) - V_2(\tau)||_{Y_1} \leq ||\beta||_{Y_2} C(T_1, T_2; \tau) \sup_{\tau} ||T_1(\tau) - T_2(\tau)|| \),

with \( C(T_1, T_2; \tau) = \hat{T}_1 e^{\hat{T}_1\tau} \int_0^\tau \int_0^s \varphi(u)duds + e^{\hat{T}_2\tau} \int_0^\tau \varphi(s)ds \).
Main results

One recovers the conditional expectation almost explicitly

- Recall the stochastic representation

\[ V^Z(t)(t, X(t)) = \mathbb{E}^Q \left[ \int_t^T e^{-r(s-t)} d(B + D)(s) \ \bigg| \ Z(t), X(t) \right] \]

- Now special case where the surplus is unchanged in transitions between states i.e., \( c^{jk}(t) - \delta^{jk}(t, x) \equiv 0 \)
- Then (2) becomes

\[ V(\tau, y) = \int_0^\tau e^{-r(\tau-s)} [G(\tau, s) \exp M(s) \beta(s)] (y) \ ds. \]

Here \( M(s) = \int_0^s T(s') \ ds' \) by componentwise integration
- The product of commuting operators \( G(\tau, s) \exp M(s) \) corresponds to the product measure \( Q \)
Main results

Pointwise sensitivities in the special case

Define $\beta'(s, y) = e^{rs} \beta(s) \exp M(s)$.

**Theorem**

Choose $p > 1$ and let $W^j(\tau, y)$ be a solution of the PDE

$$
\partial_\tau W^j = A^j(\tau) W^j - (r - \sigma_p) W^j + |T^{1-1/p} \partial_y \beta'^j(\tau, \cdot)|^p
$$

$W^j(0) = 0$,

where $\sigma_p$ is a constant depending on $A^j$. The gradient of the reserve is bounded pointwise

$$
|\partial_y V^j(\tau, y)|^p \leq W^j(\tau, y)
$$

for $(\tau, y) \in [0, T] \times \mathbb{R}$.
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Existence via operator algebra I

Proposition

Let $\theta \in [0, 1]$. Then $T$ is a bounded linear operator mapping $C^{0, \theta}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$ to itself.

Moreover there exists an evolution family $G$ which is smoothing
Existence via operator algebra II

**Proposition ([Lor11])**

Each operator $A^j$ generates an evolution system $G^j(\tau, s)$ of bounded linear operators such that for every $0 \leq \alpha \leq \gamma \leq 1$ there is a constant $c$ with

$$\| G^j(\tau, s)f \|_{C^\gamma_b(\mathbb{R})} \leq c(\tau - s)^{-1}(\gamma - \alpha)/2 \| f \|_{C^\alpha_b(\mathbb{R})}$$

for $f \in C^\alpha_b(\mathbb{R})$ and every $s \leq \tau \leq T$
Existence via operator algebra III

Idea for showing existence and uniqueness of solutions:

(i) a-priori estimates via Gronwall’s inequality

(ii) Explicit construction of a solution through a Neumann series, it converges by the a-priori estimates

(iii) Uniqueness of the solution again by Gronwall
Existence via operator algebra IV

Lemma (Gronwall’s inequality)

Suppose that for a non-negative absolutely continuous function \( \eta \) on \([0, T]\)

\[
\eta'(t) \leq \phi(t)\eta(t) + \psi(t).
\]

Then

\[
\eta(t) \leq e^{\int_0^t \phi(s) ds} \left[ \eta(0) + \int_0^t \psi(s) ds \right].
\]

Proof: A calculation shows

\[
\frac{d}{ds} \left( \eta(s) e^{-\int_0^s \phi(r) dr} \right) = e^{-\int_0^s \phi(r) dr} (\eta'(s) - \phi(s)\eta(s))
\]

\[
\leq e^{\int_0^s \phi(r) dr} \psi(s),
\]

whence the assertion.
Existence via operator algebra $V$

Let $Y_1 = C^\alpha(\mathbb{R}) \otimes \mathbb{R}^n$ and $Y_2 = C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n$. Uniform estimates yield

$$\|V(\tau)\|_{Y_1} \leq c \hat{T} \int_0^\tau \|V(s)\|_{Y_1} + c\|\beta\|_{C^{0,\alpha}([0, T] \times \mathbb{R}) \otimes \mathbb{R}^n} \phi(\tau),$$

with $\hat{T} = \sup_s \|T(s)\|$, the supremum of the operator norms of $T$ and $\phi(\tau) = \int_0^\tau e^{-r(\tau-s)} ds$. The constant $c$ comes from Proposition 6. Gronwall now implies

$$\|V(\tau)\|_{Y_1} \leq c\|\beta\|_{Y_2} \left( c \hat{T} e^{c\hat{T}} \int_0^\tau \phi(s) ds + \phi(\tau) \right). \quad (4)$$

Gives a-priori estimates: uniform norm of the reserve is bounded by global constants.
Existence via operator algebra VI

**Existence:** General approach to solving *Volterra equations* of the form

\[ u(\tau) = f(\tau) + \int_{0}^{\tau} T(s)u(s)ds, \]

cf. [Kre99]: set \( Au = \int_{0}^{\tau} T(s)u(s)ds \) and write

\[ u = Au + f \quad \text{or} \quad (I - A)u = f. \]

The idea is then to invert the operator \( I - A \) as

\[ (I - A)^{-1} = 1 + A + A^2 + \cdots . \]

Application of this Neumann series in spectral theory of Banach algebras, PDEs, etc.
Existence via operator algebra \text{VII}

Use this to find a solution for small values of \( T \) in a Neumann series with \( f(\tau) = \int_0^\tau e^{rs} \beta(s) \, ds \) as

\[
V = e^{-r\tau} \left( f + GT\#f + GT\#GT\#f + \cdots \right).
\]

under the operation

\[
(GT\#\xi)(\tau) = \int_0^\tau G(\tau, s)T(s)\xi(s) \, ds.
\]

Now iterate on the time axis with new initial conditions. This works because the a-priori estimates (4) only depend on global constants (and not on the time \( T \))
Existence via operator algebra VIII

**Uniqueness:** Let $V_1$ and $V_2$ be two solutions of the PDE. Then consider $U = V_1 - V_2$, which satisfies a linear homogeneous integral equation with initial condition $U(0) = V_1(0) - V_2(0) = 0$:

$$U(\tau) = \int_0^\tau G(\tau, s) T(s) U(s) \, ds.$$  

Gronwall then shows that $U(\tau) = 0$ for all $\tau$. 
Sketch of the proofs

Proof of the sensitivities I

This also follows from operator estimates

**Uniform estimates:** Illustration for the dependence on $T$: consider the PDEs for $V_1$ and $V_2$ for two values of the operator $T_1$ and $T_2$. By linearity

\[
V_1(\tau) - V_2(\tau) = \int_0^\tau G(\tau, s) \left( T_1(s)V_1(s) - T_2(s)V_2(s) \right) ds
\]

\[
= \int_0^\tau G(\tau, s) T_2(s) \left( V_1(s) - V_2(s) \right) ds
\]

\[
+ \int_0^\tau G(\tau, s) \left( T_1(s) - T_2(s) \right) V_1(s) ds.
\]

Then apply Gronwall twice to obtain the result.
Proof of the sensitivities II

Pointwise estimates: The basis is the

**Theorem ([KLL10])]**

For every $p > 1$ we have for all $f \in C_b^1(\mathbb{R})$ that

$$
| \left( \partial_x G^j(\tau, s)f \right)(x) |^p \leq e^{c(\tau-s)} \left( G^j(\tau, s) | \partial_x f |^p \right)(x)
$$

(5)

with $s \leq \tau$ and $x \in \mathbb{R}$. Here $c$ is a constant depending on $p$ and $A$
Proof of the sensitivities III

Application to

\[ V(\tau, y) = \int_0^\tau e^{-r(\tau-s)} [G(\tau, s) \exp M(s)\beta(s)] (y) \, ds \]

leads to an integral equation whose upper bound (5) can be translated to the PDE

\[ \partial_\tau W^j = A^j(\tau) W^j - (r - c) W^j + \left| T^{1-1/p} \partial_y \beta'^{ij}(\tau, \cdot) \right|^p \]

\[ W^j(0) = 0. \]

This is possible as \( V \) is a classical solution i.e., belongs to \( C^{1,2} \).
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Potential next steps

Polynomial processes

- Model the risky asset by polynomial processes e.g., a Lévy process.
- The operator approach can be applied formally. The operators $A_j$ become pseudo-differential operators which are defined by Fourier analysis ($a$ is the symbol of the process)

$$A u(x) = \int \int e^{i \xi (x-y)} a(x, y, \xi) u(y) dy d\xi,$$

precise structure of $a$ from Lévy-Khinchin
- Increased technical requirements and solution living in Sobolev spaces or $C^\infty$
Potential next steps II

Heat kernel methods

- Short-time asymptotic expansion of the reserve in $\tau$ around maturity $T$
- Basis is an asymptotic expansion of the Schwartz kernel of $G \sim G_0 + (\tau - s)G_1 + \cdots$, the so-called heat kernel. This is standard in differential geometry (Atiyah-Singer index theorem), quantum gravity, financial maths, ...
- Looks like

\[
V(\tau) \sim \int_0^\tau e^{-(\tau - s)r} G_0(\tau, s) \beta(s) ds + \cdots
\]
Liquidity risk

- Incorporate liquidity risk in the behaviour of the risky asset
- Leads to diffusion-degenerate nonlinear Thiele equation with error term quadratic in the spatial gradient of the reserve
- Solution given as weak solution in an $L^2$-space
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