A Representation of Excessive Functions as Expected Suprema

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Dedicated to the memory of Kazimierz Urbanik

Abstract
For a nice Markov process such as Brownian motion on a domain in $\mathbb{R}^d$, we prove a representation of excessive functions in terms of expected suprema. This is motivated by recent work of El Karoui [5] and El Karoui and Meziou [8] on the max-plus decomposition for supermartingales. Our results provide a singular analogue to the non-linear Riesz representation in El Karoui and Föllmer [6], and they extend the representation of potentials in Föllmer and Knispel [10] by clarifying the role of the boundary behavior and of the harmonic points of the given excessive function.

Key words: Markov processes, excessive functions, expected suprema

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1 Introduction
Consider a bounded superharmonic function $u$ on the open disk $S$. Such a function admits a limit $u(y)$ in almost all boundary points $y \in \partial S$ with respect to the fine topology, and we have

$$u(x) \geq \int u(y) \mu_x(dy),$$

where $\mu_x$ denotes the harmonic measure on the boundary. The right-hand side defines a harmonic function $h$ on $S$, and the difference $u - h$ can be represented as the potential of a measure on $S$. This is the classical Riesz representation of the superharmonic function $u$.

In probabilistic terms, $\mu_x$ may be viewed as the exit distribution of Brownian motion on $S$ starting in $x$, $u$ is an excessive function of the process, the fine limit can be described as a limit along Brownian paths to the boundary, and the Riesz representation takes the form

$$u(x) = E_x[\lim_{t \uparrow \zeta} u(X_t) + A_{\zeta}],$$

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where $\zeta$ denotes the first exit time from $S$ and $(A_t)_{t \geq 0}$ is the additive functional generating the potential $u - h$; cf., e.g., Blumenthal and Getoor [4].

In this paper we consider an alternative probabilistic representation of the excessive function $u$ in terms of expected suprema. We construct a function $f$ on the closure of $S$ which coincides with the boundary values of $u$ on $\partial S$ and yields the representation

$$u(x) = E_x[ \sup_{0 < t \leq \zeta} f(X_t)], \quad (1)$$

i.e.,

$$u(x) = E_x[ \sup_{0 < t < \zeta} f(X_t) \vee \lim_{t \uparrow \zeta} u(X_t)]. \quad (2)$$

Instead of Brownian motion on the unit disk, we consider a general Markov process with state space $S$ and life time $\zeta$. Under some regularity conditions we prove in section 3 that an excessive function $u$ admits a representation of the form (1) in terms of some function $f$ on $S$. Under additional conditions, the limit in (2) can be identified as a boundary value $f(X_\zeta)$ for some function $f$ on the Martin boundary of the process, and in this case (2) can also be written in the condensed form (1).

The representing function $f$ is in general not unique. In section 4 we characterize the class of representing functions in terms of a maximal and a minimal representing function. These bounds are described in potential theoretic terms. They coincide in points where the excessive function $u$ is not harmonic, the lower bound is equal to zero on the set $H$ of harmonic points, and the upper bound is constant on the connected components of $H$.

Our representation (2) of an excessive function is motivated by recent work of El Karoui and Meziou [8] and El Karoui [5] on problems of portfolio insurance. Their results involve a representation of a given supermartingale as the process of conditional expected suprema of another process. This may be viewed as a singular analogue to a general representation for semimartingales in Bank and El Karoui [1], which provides a unified solution to various representation problems arising in connection with optimal consumption choice, optimal stopping, and multi-armed bandit problems. We refer to Bank and Föllmer [2] for a survey and to the references given there, in particular to El Karoui and Karatzas [7] and Bank and Riedel [3]; see also Kaspi and Mandelbaum [11].

In the context of probabilistic potential theory such representation problems take the following form: For a given function $u$ and a given additive functional $(B_t)_{t \geq 0}$ of the underlying Markov process we want to find a function $f$ such that

$$u(x) = E_x[ \int_0^\zeta \sup_{0 < t \leq \zeta} f(X_t) \, dB_t].$$

In El Karoui and Föllmer [6] this potential theoretic problem is discussed for the smooth additive functional $B_t = t \wedge \zeta$ and for the case when $u$ has boundary behavior zero. The results are easily extended to the case where the random measure corresponding to the additive functional satisfies the regularity assumptions required in [1].
Our representation (2) corresponds to the singular case $B_t = 1_{[\zeta, \infty)}(t)$ where the random measure is given by the Dirac measure $\delta_{\zeta}$. This singular representation problem, which does not satisfy the regularity assumptions of [1], is discussed in Föllmer and Knispel [10] for the special case of a potential $u$. The purpose of the present paper is to consider a general excessive function $u$ and to clarify the impact of the boundary behavior on the representation of $u$ as an expected supremum. We concentrate on those proofs which involve explicitly the boundary behavior of $u$, and we refer to [10] whenever the argument is the same as in the case of a potential.

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2 Preliminaries

Let $(X_t)_{t \geq 0}$ be a strong Markov process with locally compact metric state space $(S, d)$, shift operators $(\theta_t)_{t \geq 0}$, and life time $\zeta$, defined on a stochastic base $(\Omega, F, (F_t)_{t \geq 0}, (P_x)_{x \in S})$ and satisfying the assumptions in [6] or [10]. In particular we assume that the excessive functions of the process are lower-semicontinuous. As a typical example, we could consider a Brownian motion on a domain $S \subset \mathbb{R}^d$.

For any measurable function $u \geq 0$ on $S$ and for any stopping time $T$ we use the notation

$$P_T u(x) := E_x[u(X_T); T < \zeta].$$

Recall that $u$ is excessive if $P_t u \leq u$ for any $t > 0$ and $\lim_{t \to 0} P_t u(x) = u(x)$ for any $x \in S$. In that case the process $(u(X_t)1_{\{t < \zeta\}})_{t \geq 0}$ is a right-continuous $P_x$-supermartingale for any $x \in S$ such that $u(x) < \infty$, and this implies the existence of

$$u_\zeta := \lim_{t \uparrow \zeta} u(X_t) \quad P_x-a.s..$$

Let us denote by $T(x)$ the class of all exit times

$$T_U := \inf\{t \geq 0 | X_t \not\in U\} \land \zeta$$

from open neighborhoods $U$ of $x \in S$, and by $T_0(x)$ the subclass of all exit times from open neighborhoods of $x$ which are relatively compact. Note that $\zeta = T_S \in T(x)$. For $T \in T(x)$ and any measurable function $u \geq 0$ we introduce the notation

$$u_T := u(X_T)1_{\{T < \zeta\}} + \lim_{t \uparrow \zeta} u(X_t)1_{\{T = \zeta\}}$$

and

$$\tilde{P}_T u(x) := E_x[u_T] = P_T u(x) + E_x[\lim_{t \uparrow \zeta} u(X_t); T = \zeta].$$
We say that a function \( u \) belongs to class (D) if for any \( x \in S \) the family \( \{ u(X_T) | T \in T_0(x) \} \) is uniformly integrable with respect to \( P_x \). Recall that an excessive function \( u \) is harmonic on \( S \) if \( P_T u(x) = u(x) \) for any \( x \in S \) and any \( T \in T_0(x) \). A harmonic function \( u \) of class (D) also satisfies \( u(x) = \hat{P}_T u(x) \) for all \( T \in T(x) \), and \( u \) is uniquely determined by its boundary behavior:

\[
    u(x) = E_x[\lim_{t \uparrow \zeta} u(X_t)] = E_x[u_{\zeta}] \quad \text{for any } x \in S. \tag{3}
\]

**Proposition 2.1** Let \( f \geq 0 \) be an upper-semicontinuous function on \( S \) and let \( \phi \geq 0 \) be \( \mathcal{F} \)-measurable such that \( \phi = \phi \circ \theta_T \) \( P_x \)-a.s. for any \( x \in S \) and any \( T \in T_0(x) \). Then the function \( u \) on \( S \) defined by the expected suprema

\[
    u(x) := E_x[\sup_{0 < t < \zeta} f(X_t) \vee \phi] \tag{4}
\]

is excessive, hence lower-semicontinuous. Moreover, \( u \) belongs to class (D) if and only if \( u \) is finite on \( S \). In this case \( u \) has the boundary behavior

\[
    u_{\zeta} = \lim_{t \uparrow \zeta} f(X_t) \vee \phi = f_{\zeta} \vee \phi \quad P_x \text{-a.s.,} \tag{5}
\]

and \( u \) admits a representation (2), i.e., a representation (4) with \( \phi = u_{\zeta} \).

**Proof.** It follows as in [10] that \( u \) is an excessive function. If \( u(x) < \infty \) then

\[
    \sup_{0 < t < \zeta} f(X_t) \vee \phi \in L^1(P_x).
\]

Thus \( \{ u(X_T) | T \in T_0(x) \} \) is uniformly integrable with respect to \( P_x \), since

\[
    0 \leq u(X_T) = E_x[\sup_{T < t < \zeta} f(X_t) \vee (\phi \circ \theta_T)|\mathcal{F}_t] \leq E_x[\sup_{0 < t < \zeta} f(X_t) \vee \phi|\mathcal{F}_T]
\]

for all \( T \in T_0(x) \). Conversely, if \( u \) belongs to class (D) then \( u \) is finite on \( S \) since by lower-semicontinuity

\[
    u(x) \leq E_x[\lim_{n \uparrow \infty} u(X_{T_{\epsilon_n}})] \leq \lim_{n \uparrow \infty} E_x[u(X_{T_{\epsilon_n}})] < \infty,
\]

for \( \epsilon_n \downarrow 0 \), where \( T_{\epsilon_n} \in T_0(x) \) denotes the exit time from the open ball \( U_{\epsilon_n}(x) \).

In order to verify (5), we take a sequence \( (U_n)_{n \in \mathbb{N}} \) of relatively compact open neighborhoods of \( x \) increasing to \( S \) and denote by \( T_n \) the exit time from \( U_n \). Since \( u \) is excessive and finite on \( S \) we conclude that

\[
    \lim_{t \uparrow \zeta} f(X_t) \vee \phi = \lim_{n \uparrow \infty} \sup_{T_n < t < \zeta} f(X_t) \vee (\phi \circ \theta_{T_n})
    = \lim_{n \uparrow \infty} E_x[\sup_{T_n < t < \zeta} f(X_t) \vee (\phi \circ \theta_{T_n})|\mathcal{F}_{T_n}]
    = \lim_{n \uparrow \infty} u(X_{T_n}) = u_{\zeta} \quad P_x \text{-a.s.,}
\]

where the second identity follows from a martingale convergence argument.

In view of (5) we have

\[
    \{ \phi \leq \sup_{0 < t < \zeta} f(X_t) \} = \{ u_{\zeta} \leq \sup_{0 < t < \zeta} f(X_t) \} \quad P_x \text{-a.s.}
\]
and \( \phi = u_\zeta \) on \( \{ \phi > \sup_{0 < t < \zeta} f(X_t) \} \) \( \mathbb{P}_x \)-a.s.. Thus we can write
\[
\begin{align*}
u(x) &= E_x[ \sup_{0 < t < \zeta} f(X_t); \phi \leq \sup_{0 < t < \zeta} f(X_t)] + E_x[\phi; \phi > \sup_{0 < t < \zeta} f(X_t)] \\
&= E_x[ \sup_{0 < t < \zeta} f(X_t); u_\zeta \leq \sup_{0 < t < \zeta} f(X_t)] + E_x[u_\zeta; u_\zeta > \sup_{0 < t < \zeta} f(X_t)] \\
&= E_x[ \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta].
\end{align*}
\]

In the next section we show that, conversely, any excessive function \( u \) of class (D) admits a representation of the form (2), where \( f \) is some upper-semicontinuous function on \( S \).

3 Construction of a representing function

Let \( u \geq 0 \) be an excessive function of class (D). In order to avoid additional technical difficulties, we also assume that \( u \) is continuous. For convenience we introduce the notation \( u^c := u \vee c \).

Consider the family of optimal stopping problems
\[
Ru^c(x) := \sup_{T \in \mathcal{T}_0(x)} E_x[u^c(X_T)]
\]
for \( c \geq 0 \) and \( x \in S \). It is well known that the value function \( Ru^c \) of the optimal stopping problem (6) can be characterized as the smallest excessive function dominating \( u^c \). In particular, \( Ru^c \) is lower-semicontinuous. Moreover,
\[
Ru^c(x) \geq E_x[u^c(X_T); T < \zeta] + E_x[\lim_{t \uparrow \zeta} u^c(X_t); T = \zeta] = \tilde{P}_T u^c(x)
\]
for any stopping time \( T \leq \zeta \), and equality holds for the first entrance time into the closed set \( \{ Ru^c = u^c \} \); cf. for example the proof of Lemma 4.1 in [6].

The following lemma can be verified by a straightforward modification of the arguments in [10]:

**Lemma 3.1**

1) For any \( x \in S \), \( Ru^c(x) \) is increasing, convex and Lipschitz-continuous in \( c \), and
\[
\lim_{c \downarrow \infty} (Ru^c(x) - c) = 0.
\]

2) For any \( c \geq 0 \),
\[
Ru^c(x) = E_x[u^c_{D^c}] = \tilde{P}_{D^c} u^c(x),
\]
where \( D^c := \inf \{ t \geq 0 \mid Ru^c(X_t) = u(X_t) \} \) is the first entrance time into the closed set \( \{ Ru^c = u \} \). Moreover, the map \( c \mapsto D^c \) is increasing and \( \mathbb{P}_x \)-a.s. left-continuous.

Since the function \( c \mapsto Ru^c(x) \) is convex, it is almost everywhere differentiable. The following identification of the derivatives is similar to Lemma 3.2 of [10].

**Lemma 3.2** The left-hand derivative \( \partial^- Ru^c(x) \) of \( Ru^c(x) \) with respect to \( c > 0 \) is given by
\[
\partial^- Ru^c(x) = \mathbb{P}_x[u_\zeta < c, D^c = \zeta].
\]
Proof. For any $0 \leq a < c$, the representation (9) for the parameter $c$ combined with the inequality (7) for the parameter $a$ and for the stopping time $T = D^c$ implies

$$Ru^c(x) - Ru^a(x) \leq E_x[u^c(X_{D^c}) - u^a(X_{D^c}); D^c < \zeta] + E_x[u^c - u^a; D^c = \zeta].$$

Since

$$u(X_{D^c}) = Ru^c(X_{D^c}) \geq c > a \quad \text{on } \{D^c < \zeta\}$$

and $u^c - u^a \leq (c - a)1_{\{u^c < c\}}$, the previous estimate simplifies to

$$Ru^c(x) - Ru^a(x) \leq E_x[u^c(X_{D^c}) - u^a(X_{D^c}); D^c < \zeta] + E_x[u^c - u^a; D^c = \zeta].$$

This shows $\partial^- Ru^c(x) \leq P_x[u^c < c, D^c = \zeta]$. In order to prove the converse inequality, we use the estimate

$$Ru^c(x) - Ru^a(x) \geq (c - a)P_x[u^c < c, D^a = \zeta]$$

obtained by reversing the role of $a$ and $c$ in the preceding argument. This implies

$$\partial^- Ru^c(x) \geq \lim_{a\uparrow c} P_x[u^c < c, D^a = \zeta] = P_x[u^c < c, D^c = \zeta]$$

since $\bigcup_{a < c} \{D^a = \zeta\} = \{D^c = \zeta\}$ on $\{u^c < c\}$, due to the Lipschitz-continuity of $Ru^c(x)$ in $c$. $\square$

Let us now introduce the function $f^*$ defined by

$$f^*(x) := \sup\{c| x \in \{Ru^c = u\}\}$$ (10)

for any $x \in S$. Note that $f^*(x) \geq c$ is equivalent to $Ru^c(x) = u(x)$ due to the continuity of $Ru^c(x)$ in $c$. It follows as in [10], Lemma 3.3, that the function $f^*$ is upper-semicontinuous and satisfies $0 \leq f^* \leq u$.

We are now ready to derive a representation of the value functions $Ru^c$ in terms of the function $f^*$. In the special case of a potential $u$, where $u_\zeta = 0$ and $u^c_\zeta = c P_x$-a.s., our representation (11) reduces to Theorem 3.1 of [10].

**Theorem 3.1** For any $c \geq 0$ and any $x \in S$,

$$Ru^c(x) = E_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u^c_\zeta] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u^c_\zeta].$$ (11)

**Proof.** By Lemma 3.2 and (8) we get

$$Ru^c(x) - c = \int_c^\infty -\frac{\partial}{\partial a} (Ru^a(x) - a) \ da = \int_c^\infty (1 - P_x[u^c < a, D^a = \zeta]) \ da.$$

Since

$$\{D^{c+\epsilon} < \zeta\} \subseteq \{\sup_{0 \leq t < \zeta} f^*(X_t) > c\} \subseteq \{D^c < \zeta\}$$
for any $c \geq 0$ and for any $\epsilon > 0$,

$$Ru^c(x) - c = \int_c^\infty (1 - P_x[u_\zeta < \alpha, D^\alpha = \zeta]) \, d\alpha$$

$$\geq \int_c^\infty (1 - P_x[u_\zeta \leq \alpha, \sup_{0 \leq t < \zeta} f^*(X_t) \leq \alpha]) \, d\alpha$$

$$\geq \int_c^\infty (1 - P_x[u_\zeta < \alpha + \epsilon, D^{\alpha+\epsilon} = \zeta]) \, d\alpha$$

$$= Ru^{c+\epsilon}(x) - (c + \epsilon).$$

By continuity of $c \mapsto Ru^c$ we obtain

$$Ru^c(x) - c \geq \int_c^\infty (1 - P_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta \leq \alpha]) \, d\alpha$$

$$\geq \lim_{\epsilon \downarrow 0} Ru^{c+\epsilon}(x) - (c + \epsilon) = Ru^c(x) - c,$$

hence

$$Ru^c(x) = \int_c^\infty P_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta > \alpha] \, d\alpha + c$$

$$= E_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta - (\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta) \wedge c + c]$$

$$= E_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta].$$

Moreover, we can conclude that

$$Ru^c(x) = \lim_{t \downarrow 0} P_t(Ru^c)(x) = \lim_{t \downarrow 0} E_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta; t < \zeta] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta]$$

since $Ru^c$ is excessive, i.e., $Ru^c(x)$ also admits the second representation in equation (11). □

As a corollary we see that $f^*$ is a representing function for $u$.

**Corollary 3.1** The excessive function $u$ admits the representations

$$u(x) = E_x[\sup_{0 \leq t < \zeta} f^*(X_t) \vee u_\zeta] = E_x[\sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta] \tag{12}$$

in terms of the upper-semicontinuous function $f^* \geq 0$ defined by (10). Moreover,

$$f^*(x) \leq \sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta \quad P_x - a.s.\text{ for any } x \in S.$$
Remark 3.1 Under additional regularity conditions, the underlying Markov process admits a Martin boundary $\partial S$, i.e., a compactification of the state space such that $\lim_{t\uparrow \zeta} u(X_t)$ can be identified with the values $f(X_\zeta)$ for a suitable continuation of the function $f$ to the Martin boundary; cf., e.g., [9], (4.12) and (5.7). In such a situation the general representation (12) may be written in the condensed form (1).

Corollary 3.1 shows that $u$ admits a representing function which is regular in the following sense:

**Definition 3.1** Let us say that a nonnegative function $f$ on $S$ is regular with respect to $u$ if it is upper-semicontinuous and satisfies the condition

$$f(x) \leq \sup_{0 \leq t < \zeta} f(X_t) \lor u_\zeta \quad P_x - \text{a.s.}$$

for any $x \in S$.

Note that a regular function $f$ also satisfies the inequality

$$f(X_T) \leq \sup_{T < t < \zeta} f(X_t) \lor u_\zeta \quad P_x - \text{a.s. on } \{T < \zeta\}$$

for any stopping time $T$, due to the strong Markov property.

4 The minimal and the maximal representation

Let us first derive an alternative description of the representing function $f^*$ in terms of the given excessive function $u$. To this end, we introduce the superadditive operator

$$Du(x) := \inf\{c \geq 0 | \exists T \in T(x) : \tilde{P}_T u^c(x) > u(x)\}.$$ 

**Proposition 4.1** The functions $f^*$ and $Du$ coincide. In particular, $x \mapsto Du(x)$ is regular with respect to $u$.

**Proof.** Recall that $f^*(x) \geq c$ is equivalent to $Ru^c(x) = u(x)$. Thus $f^*(x) \geq c$ yields

$$u(x) = Ru^c(x) \geq \tilde{P}_T u^c(x)$$

for any $T \in T(x)$ due to (7). This amounts to $Du(x) \geq c$, and so we obtain $f^*(x) \leq Du(x)$.

In order to prove the converse inequality, we take $c > f^*(x)$ and define $T_c \in T(x)$ as the first exit time from the open neighborhood $\{f^* < c\}$ of $x$. Then

$$u(x) < Ru^c(x) = Ex \left[ \sup_{0 \leq t < \zeta} f^*(X_t) \lor u_\zeta \right]$$

$$= Ex \left[ \sup_{T_c \leq t < \zeta} f^*(X_t) \lor u_\zeta; T_c < \zeta \right] + Ex[u_\zeta; T_c = \zeta]$$

$$= Ex \left[ (Ex_{T_c} \left[ \sup_{0 \leq t < \zeta} f^*(X_t) \lor u_\zeta \right] \lor c; T_c < \zeta \right] + Ex[u_\zeta; T_c = \zeta]$$

$$= Ex[u^c(X_{T_c}); T_c < \zeta] + Ex[u_\zeta; T_c = \zeta] = \tilde{P}_{T_c} u^c(x),$$

hence $Du(x) \leq c$. This shows $Du(x) \leq f^*(x)$. $\Box$
Remark 4.1  A closer look at the preceding proof shows that
\[ Du(x) = \inf\{ c \geq 0 \exists T \in T(x) : u(x) - P_T u(x) < E_x[u_\zeta; T = \zeta] \}. \]

For any potential \( u \) of class (D) we have \( u_\zeta = 0 \ P_x \)-a.s., and so we get
\[ Du(x) = \inf\{ u(x) - P_T u(x) \over P_x[T = \zeta] \}, \]
where the infimum is taken over all exit times \( T \) from open neighborhoods of \( x \) such that \( P_x[T = \zeta] > 0 \). Thus our general representation in Corollary 3.1 contains as a special case the representation of a potential of class (D) given in [10].

We are now going to identify the maximal and the minimal representing function for the given excessive function \( u \).

Theorem 4.1  Suppose that \( u \) admits the representation
\[ u(x) = E_x[ \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta] \]
for any \( x \in S \), where \( f \) is regular with respect to \( u \) on \( S \). Then \( f \) satisfies the bounds
\[ f_* \leq f \leq f^* = Du, \]
where the function \( f_* \) is defined by
\[ f_*(x) := \inf\{ c \geq 0 \exists T \in T(x) : \tilde{P}_T u^c(x) \geq u(x) \} \]
for any \( x \in S \).

Proof. Let us first show that \( f \leq f^* = Du \). If \( f(x) \geq c \) then we get for any \( T \in T(x) \)
\[ u(x) = E_x[ \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta] \geq E_x[ \sup_{T < t < \zeta} f(X_t) \vee u_\zeta; T < \zeta] + E_x[u_\zeta; T = \zeta] \]
\[ \geq E_x[ E_x[ \sup_{T < t < \zeta} f(X_t) \vee u_\zeta; T < \zeta] + E_x[u_\zeta; T = \zeta] = \tilde{P}_T u^c(x) \]
due to our assumption (13) on \( f \) and Jensen’s inequality. Thus \( Du(x) \geq c \), and this yields \( f(x) \leq Du(x) \). In order to verify the lower bound, take \( c > f(x) \) and let \( T_c \in T(x) \) denote the first exit time from \( \{ f < c \} \). Since
\[ c \leq f(X_{T_c}) \leq \sup_{T_c < t < \zeta} f(X_t) \vee u_\zeta = \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta \ P_\zeta \text{-a.s. on } \{ T_c < \zeta \} \]
due to property (14) of \( f \), we obtain
\[ \tilde{P}_{T_c} u^c(x) = E_x[u^c(X_{T_c}); T_c < \zeta] + E_x[u_\zeta; T_c = \zeta] \]
\[ = E_x[ E_x[ \sup_{T_c < t < \zeta} f(X_t) \vee u_\zeta; T_c < \zeta] + E_x[u_\zeta; T_c = \zeta] \]
\[ = E_x[ \sup_{T_c < t < \zeta} f(X_t) \vee u_\zeta; T_c = \zeta] + E_x[ \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta; T_c = \zeta] \]
\[ \geq E_x[ \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta] = u(x), \]
hence \( c \geq f_*(x) \). This implies \( f_*(x) \leq f(x) \).

The following example shows that the representing function may not be unique, and that it is in general not possible to drop the limit \( u_\zeta \) in the representation (2).
Example 4.1 Let \((X_t)_{t \geq 0}\) be a Brownian motion on the interval \(S = (0, 3)\). Then the function \(u\) defined by
\[
   u(x) = \begin{cases} 
   x & , x \in (0, 1) \\
   \frac{1}{2}x + \frac{1}{2}, & x \in [1, 2] \\
   \frac{1}{2}x + 1, & x \in (2, 3) 
   \end{cases}
\]
is concave on \(S\), hence excessive. Here the maximal representing function \(f^*\) takes the form
\[
   f^*(x) = \frac{1}{2}1_{[1,2]}(x) + 1_{[2,3]}(x),
\]
and \(f_*\) is given by \(f_*(x) = \frac{1}{2}1_{[1]}(x) + 1_{[2]}(x)\). In particular we get for any \(x \in (2, 3)\)
\[
   u(x) > E_x[\sup_{0<t<\zeta} f^*(X_t)].
\]
This shows that we have to include \(u_\zeta\) into the representation of \(u\). Moreover, for any \(x \in S\)
\[
   \sup_{0<t<\zeta} f_*(X_t) \lor u_\zeta = \sup_{0<t<\zeta} f^*(X_t) \lor u_\zeta \geq f^*(x) \geq f_*(x) \quad P_x \cdot \text{a.s.,}
\]
and so \(f_*\) is a regular representing function for \(u\). In particular, the representing function
is not unique.

We are now going to derive an alternative description of \(f_*\) which will allow us to identify
\(f_*\) as the minimal representing function for \(u\).

Definition 4.1 Let us say that a point \(x_0 \in S\) is harmonic for \(u\) if the mean-value property
\[
   u(x_0) = E_{x_0}[u(X_{T_\epsilon})]
\]
holds for \(x_0\) and for some \(\epsilon > 0\), where \(T_\epsilon\) denotes the first exit time from the ball \(U_\epsilon(x_0)\).
We denote by \(H\) the set of all points in \(S\) which are harmonic with respect to \(u\).

Under the regularity assumptions of [10], the set \(H\) coincides with the set of all points \(x_0 \in S\)
such that \(u\) is harmonic in some open neighborhood \(G\) of \(x_0\), i.e., the mean-value property
\[
   u(x) = E_x[u(X_{T_{\epsilon}(x)})]
\]
holds for all \(x \in G\) and all \(\epsilon > 0\) such that \(\overline{U_\epsilon(x)} \subset G\); cf. Lemma 4.1 in [10]. In particular,
\(H\) is an open set.

The following proposition extends Proposition 4.1 in [10] from potentials to general excessive functions.

Proposition 4.2 For any \(x \in S\),
\[
   f_*(x) = f^*(x)1_{H^c}(x). \tag{15}
\]
In particular, \(f_*\) is upper-semicontinuous.
For \( x \in H \) there exists \( \epsilon > 0 \) such that \( \overline{U_\epsilon(x)} \subset S \) and \( u(x) = E_x[u(X_{T_{U_\epsilon(x)}})] = \tilde{P}_{T_{U_\epsilon(x)}} u^0(x) \), and this implies \( f_*(x) = 0 \). Now suppose that \( x \in H^c \), i.e., \( u \) is not harmonic in \( x \). Let us first prove that
\[
\tilde{P}_T u(x) < u(x) \quad \text{for all } T \in T(x).
\] (16)
Indeed, if \( T \) is the first exit time from some open neighborhood \( G \) of \( x \) then
\[
\tilde{P}_T u(x) = E_x[E_{X_{T_{U_\epsilon(x)}}} [u(X_T); T < \zeta] + E_{X_{T_{U_\epsilon(x)}}} [u_\zeta; T = \zeta]] \\
\leq E_x[R u^0(X_{T_{U_\epsilon(x)}})] = E_x[u(X_{T_{U_\epsilon(x)}})] < u(x)
\]
for any \( \epsilon > 0 \) such that \( \overline{U_\epsilon(x)} \subseteq G \). In view of Theorem 4.1 we have to show \( f_*(x) \geq f^*(x) \), and we may assume \( f^*(x) > 0 \). Choose \( c > 0 \) such that \( f^*(x) > c \). Then there exists \( \epsilon > 0 \) such that \( Ru^{c+\epsilon}(x) = u(x) \), i.e.,
\[
\tilde{P}_T u^{c+\epsilon}(x) \leq u(x)
\] (17)
for any \( T \in T(x) \) in view of (7). Fix \( \delta \in (0, \epsilon) \) and \( T \in T(x) \). If
\[
P_x[u(X_T)] \leq c + \delta; T < \zeta] + P_x[u_\zeta \leq c + \delta; T = \zeta] > 0
\]
we get the estimate
\[
\tilde{P}_T u^{c+\delta}(x) = E_x[u^{c+\delta}(X_T); T < \zeta] + E_x[u^{c+\delta}_\zeta; T = \zeta] < \tilde{P}_T u^{c+\epsilon}(x) \leq u(x).
\]
On the other hand, if \( P_x[u(X_T)] \leq c + \delta; T < \zeta] = P_x[u_\zeta \leq c + \delta; T = \zeta] = 0 \) then
\[
\tilde{P}_T u^{c+\delta}(x) = E_x[u(X_T); T < \zeta] + E_x[u_\zeta; T = \zeta] = \tilde{P}_T u(x) < u(x)
\]
due to (16). Thus we obtain \( u(x) > \tilde{P}_T u^{c+\delta}(x) \) for any \( T \in T(x) \), hence \( f_*(x) \geq c + \delta \). This concludes the proof of (15). Upper-semicontinuity of \( f_* \) follows immediately since \( f^* \) is upper-semicontinuous and \( H^c \) is closed. \( \square \)

Our next purpose is to show that \( f^* \) is constant on connected components of \( H \).

**Proposition 4.3** For any \( x \in H \),
\[
f^*(x) = \operatorname{ess}
\inf_{P_x} f^*_T,
\] (18)
where \( T \) denotes the first exit time from the maximal connected neighborhood \( H(x) \subseteq H \) of \( x \). In particular, \( f^* \) is constant on \( H(x) \).

**Proof.** 1) Let us first show that for a connected open set \( U \subseteq S \) and for any \( x, y \in U \), the measures \( P_x \) and \( P_y \) are equivalent on the \( \sigma \)-field describing the exit behavior from \( U \):
\[
P_x \approx P_y \quad \text{on } \tilde{F}_U := \sigma \{ g_{T_{U}}, | g \text{ measurable on } S \}.
\] (19)
Indeed, any \( A \in \tilde{F}_U \) satisfies \( 1_A \circ \theta_{T_{U}} = 1_A \) if \( T_{U} \) denotes the exit time from some neighborhood \( U_\epsilon(x) \) such that \( \overline{U_\epsilon(x)} \subseteq U \). Thus
\[
P_x[A] = E_x[1_A \circ \theta_{T_{U}}] = \int P_y[A] \mu_x(dz),
\]
where \( \mu_{x,\epsilon} \) is the exit distribution from \( U(x) \). Since \( \mu_{x,\epsilon} \approx \mu_{y,\epsilon} \) by assumption \( \textbf{A3} \) of [10], we obtain \( P_x \approx P_y \) on \( \tilde{F}_U \) for any \( y \in U(x) \). For arbitrary \( y \in U \) we can choose \( x_0, \ldots, x_n \) and \( \epsilon_1, \ldots, \epsilon_n \) such that \( x_0 = x, x_n = y, x_k \in U_{\epsilon_k}(x_{k-1}) \) and \( U_{\epsilon_k}(x_{k-1}) \subset U \). Hence \( P_{x_k} \approx P_{x_k-1} \) on \( \tilde{F}_U \), and this yields (19).

2) For \( x \in H \) let \( c(x) \) be the right-hand side of equation (18). In order to verify \( f^*(x) \leq c(x) \), we take a sequence of relatively compact open neighborhoods \( (U_n(x))_{n \in \mathbb{N}} \) of \( x \) increasing to \( H(x) \) and denote by \( T_n \) the first exit time from \( U_n(x) \). Since \( f^* \) is upper-semicontinuous on \( S \), we get the estimate

\[
\lim_{n \to \infty} f^*(X_{T_n}) \leq f^*(X_T)1_{\{T < \zeta\}} + \lim_{n \to \infty} f^*(X_t)1_{\{T = \zeta\}} = f^*_T \quad P_x \text{-a.s.,}
\]

hence \( P_x[\lim_{n \to \infty} f^*(X_{T_n}) < c] > 0 \) for any \( c > c(x) \). Thus, there exists \( n_0 \) such that \( P_x[\tilde{R}_u^c(X_{T_{n_0}}) > u(X_{T_{n_0}})] = P_x[f^*(X_{T_{n_0}}) < c] > 0 \), and this implies

\[
u(x) = E_x[u(X_{T_{n_0}})] < E_x[\tilde{R}_u^c(X_{T_{n_0}})] \leq \tilde{R}_u^c(x)
\]

since \( \tilde{R}_u^c \) is excessive. But this amounts to \( f^*(x) < c \), and taking the limit \( c \downarrow c(x) \) yields \( f^*(x) \leq c(x) \).

3) In order to prove the converse inequality, we use the fact that for any \( c < c(x) \)

\[
E_x[u^c(X_{\tilde{T}})] \leq u(x) \quad \text{for all } \tilde{T} \in T_0(x),
\]

which is equivalent to \( \tilde{R}_u^c(x) = u(x) \). Thus we get \( f^*(x) \geq c \) for all \( c < c(x) \), hence \( f^*(x) = c(x) \) in view of 2). Since \( c(x) = c(y) \) for any \( y \in H(x) \) due to (19), we see that \( f^* \) is constant on \( H(x) \).

It remains to verify (20). To this end, note that for any \( y \in H(x) \) we have \( c < c(x) = c(y) \leq f^*_T \quad P_y \text{-a.s.} \) due to (19). Thus, \( f^*(X_T) > c P_y \text{-a.s.} \) on \( \{T < \zeta\} \) for any \( y \in H(x) \), and this yields

\[
u^c(X_T) \leq \tilde{R}_u^c(X_T) = u(X_T) \quad P_y \text{-a.s.} \quad \text{on } \{T < \zeta\}.
\]

Moreover, we get \( c < f^*_T \leq u_\zeta \quad P_y \text{-a.s.} \) on \( \{T = \zeta\} \). Let us now fix \( \tilde{T} \in T_0(x) \). Since \( X_{\tilde{T}} \in H(x) \) on \( \{\tilde{T} < T\} \), we can conclude that

\[
E_x[u^c(X_{\tilde{T}});T < \tilde{T}] = E_x[\tilde{P}_{\tilde{T}} u(X_{\tilde{T}}) \vee c;T < \tilde{T}] \\
\leq E_x[E_{X_{\tilde{T}}}[u^c(X_T);T < \zeta] + E_{X_{\tilde{T}}}[u_\zeta;T = \zeta];\tilde{T} < T] \\
= E_x[E_{X_{\tilde{T}}}[u(X_T);T < \zeta] + E_{X_{\tilde{T}}}[u_\zeta;T = \zeta];\tilde{T} < T] \\
= E_x[u_T;\tilde{T} < T].
\]

(21)

On the other hand, we have \( \{T \leq \tilde{T}\} \subseteq \{T < \zeta\} \), and by the \( P_x \)-supermartingale property of \( (\tilde{R}_u^c(X_t)1_{\{t < \zeta\}})_{t \geq 0} \) we get the estimate

\[
E_x[u(X_{\tilde{T}});\tilde{T} \geq T] \leq E_x[\tilde{R}_u^c(X_{\tilde{T}});\tilde{T} \geq T] \leq E_x[\tilde{R}_u^c(X_T);\tilde{T} \geq T] \\
= E_x[u(X_T);\tilde{T} \geq T] = E_x[u_T;\tilde{T} \geq T],
\]

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where the first equality follows from \( f^*(X_T) \geq c(x) > c \) \( P_x \)-a.s. on \( \{ T < \zeta \} \). In combination with (21) this yields
\[
E_x[u^c(X_T)] \leq E_x[u_T] = u(x).
\]

**Remark 4.2** A point \( x \in S \) is harmonic with respect to \( u \) if and only if there exists \( \epsilon > 0 \) such that \( f^* \) is constant on \( U_\epsilon(x) \subset S \). Indeed, Proposition 4.3 shows that this condition is necessary. Conversely, take \( x \in H^c \) and assume that there exists \( \epsilon > 0 \) such that \( f^* \) is constant on \( U_\epsilon(x) \subset S \). Then the exit time \( T := T_{U_\epsilon(x)} \) satisfies
\[
\hat{P}_T u(x) = E_x[u(X_T)] = E_x[\sup_{T < t < \zeta} f^*(X_t) \vee u_\zeta] = E_x[f^*(X_t) \vee u_\zeta] = u(x)
\]
in contradiction to (16).

Our next goal is to show that \( f_* \) is the minimal representing function for \( u \).

**Theorem 4.2** Let \( f \) be an upper-semicontinuous function on \( S \) such that \( f_* \leq f \leq f^* \). Then \( f \) is a regular representing function for \( u \). In particular we obtain the representation
\[
u(x) = E_x[\sup_{0 < t < \zeta} f_*(X_t) \vee u_\zeta],
\]
and \( f_* \) is the minimal regular function yielding a representation of \( u \).

**Proof.** Let us show that
\[
\sup_{0 < t < \zeta} f_*(X_t) \vee u_\zeta = \sup_{0 < t < \zeta} f(X_t) \vee u_\zeta = \sup_{0 < t < \zeta} f^*(X_t) \vee u_\zeta \quad P_x \text{-a.s.}
\]
for any \( x \in S \). To this end, suppose first that \( x \in H \). We denote by \( T_c \) the exit time from the open set \( \{ f^* < c \} \). Since \( 0 \leq f_* \leq f \leq f^* \), it is enough to show that for fixed \( c \geq f^*(x) \)
\[
\sup_{0 < t < \zeta} f_*(X_t) \vee u_\zeta \geq c \quad P_x \text{-a.s. on } \{ T_c < \zeta \}.
\]
By (15) we see that
\[
\sup_{0 < t < \zeta} f_*(X_t) \geq f_*(X_{T_c}) = f^*(X_{T_c}) \geq c \quad P_x \text{-a.s. on } \{ T_c < \zeta, X_{T_c} \in H^c \}.
\]
On the set \( A := \{ T_c < \zeta, X_{T_c} \in H \} \) we use the inequality
\[
f^*(X_{T_c}) \leq f^*_T \quad P_x \text{-a.s. on } A
\]
for \( T := T_c + T_H \circ \theta_{T_c} \), which follows from Proposition 4.3 combined with the strong Markov property. Using (15) and (24) we obtain
\[
\sup_{0 < t < \zeta} f_*(X_t) \geq f_*(X_T) = f^*(X_T) \geq f^*(X_{T_c}) \geq c \quad P_x \text{-a.s. on } A \cap \{ T < \zeta \}
\]
and
\[
u_\zeta \geq f^*_c \geq f^*(X_{T_c}) \geq c \quad P_x \text{-a.s. on } A \cap \{ T = \zeta \},
\]
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hence $\sup_{0 < t < \zeta} f_s(x_t) \vee u_\zeta \geq c \mathbb{P}_x$-a.s. on $\mathcal{A}$. This concludes the proof of (23) for $x \in H$, and so (22) holds for any $x \in H$. In particular, we have

$$\sup_{\tilde{T} < t < \zeta} f_s(x_t) \vee u_\zeta = \sup_{\tilde{T} < t < \zeta} f^*(x_t) \vee u_\zeta \quad \mathbb{P}_x\text{-a.s. on } \{	ilde{T} < \zeta, X_{\tilde{T}} \in H\}$$

(25)

for any stopping time $\tilde{T}$, due to the strong Markov property.

Let us now fix $x \in H^c$ and denote by $\tilde{T}$ the first exit time from $H^c$. Since the functions $f_s$ and $f^*$ coincide on $H^c$ due to Proposition 4.2, the identity (22) follows immediately on the set $\{\tilde{T} = \zeta\}$. On the other hand, using again Proposition 4.2, we get

$$\sup_{0 < t < \zeta} f^*(x_t) \vee u_\zeta = \sup_{0 < t \leq \tilde{T}} f^*(x_t) \vee \sup_{\tilde{T} < t < \zeta} f^*(x_t) \vee u_\zeta$$

(26)

$$= \sup_{0 < t \leq \tilde{T}} f_s(x_t) \vee \sup_{\tilde{T} < t < \zeta} f^*(x_t) \vee u_\zeta \quad \mathbb{P}_x\text{-a.s. on } \{	ilde{T} < \zeta\}.$$

By definition of $\tilde{T}$, on $\{	ilde{T} < \zeta\}$ there exists a sequence of stopping times $\tilde{T} < T_n < \zeta, n \in \mathbb{N}$, decreasing to $\tilde{T}$ such that $X_{T_n} \in H$. Thus,

$$\sup_{\tilde{T} < t < \zeta} f^*(x_t) \vee u_\zeta = \lim_{n \to \infty} \sup_{T_n < t < \zeta} f^*(x_t) \vee u_\zeta$$

$$= \lim_{n \to \infty} \sup_{T_n < t < \zeta} f_s(x_t) \vee u_\zeta$$

$$= \sup_{\tilde{T} < t < \zeta} f_s(x_t) \vee u_\zeta \quad \mathbb{P}_x\text{-a.s. on } \{	ilde{T} < \zeta\}$$

due to (25). Combined with (26) this yields (22) on $\{	ilde{T} < \zeta\}$. Thus we have shown that (22) holds as well for any $x \in H^c$.

In particular, $f$ is a representing function for $u$. Moreover,

$$f(x) \leq f^*(x) \leq \sup_{0 < t < \zeta} f^*(x_t) \vee u_\zeta = \sup_{0 < t < \zeta} f(x_t) \vee u_\zeta \quad \mathbb{P}_x\text{-a.s.}$$

for any $x \in S$ due to (22), and so $f$ is a regular function on $S$ with respect to $u$. In view of Theorem 4.1 we see that $f_s$ is the minimal regular representing function for $u$. \hfill $\Box$

**Remark 4.3** Suppose that $u$ admits a representation of the form

$$u(x) = E_x[\sup_{0 < t < \zeta} f(X_t)]$$

(27)

for all $x \in S$ and for some regular function $f$ on $S$. Then $f$ satisfies the bounds $f_s \leq f \leq f^*$, due to Theorem 4.1 combined with Proposition 2.1 for $\phi = 0$. Clearly such a reduced representation, which does not involve explicitly the boundary behavior of $u$, holds if and only if $u_\zeta \leq \sup_{0 < t < \zeta} f(X_t) \quad \mathbb{P}_x\text{-a.s.}$. In particular, this is the case for a potential $u$ where $u_\zeta = 0$, in accordance with the results in [10]. Example 4.1 shows that a reduced representation (27) is not possible in general. If $u$ is harmonic on $S$, (27) would in fact imply that $u$ is constant on $S$. Indeed, harmonicity of $u$ on $S$ implies that $f^* = c$ on $S$ for some constant $c$ due to Proposition 4.3, hence

$$E_x[\sup_{0 < t < \zeta} f(X_t)] \leq c \leq E_x[u_\zeta] = u(x)$$

due to $\mathbb{P}_x\text{-a.s.}$ on $S$ and so (27) would imply $u(x) = c$ for all $x \in S$.  

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References


