Inflation and Correlation in Claims Run-Off Triangles

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Agenda

- outstanding loss liabilities and claims reserving
- log-normal chain-ladder (CL) model of Hertig [1]
- dependence and aggregation: multivariate log-normal CL model

Joint work with

▷ Michael Merz (University of Hamburg)

▷ Enkelejd Hashorva (University of Lausanne)

Claims development triangle

a.y.	development year j																
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1994	13'109	7'246	982	706	358	257	339	161	334	172	35	205	56	32	2	7	1
1995	14'457	7'581	589	487	124	74	128	50	474	12	72	63	141	286	2	10	
1996	16'075	6'597	1'081	299	154	551	29	21	16	65	98	415	280	24	27		
1997	15'682	7'782	1'001	587	477	179	44	18	65	240	7	64	4	17			
1998	16'551	7'155	921	946	473	69	168	198	220	17	6	4	7				
1999	15'439	8'357	1'070	451	822	15	21	30	559	54	18	123					
2000	14'629	7'016	1'181	773	1'393	442	42	73	55	105	14						
2001	17'585	8'703	1'335	316	396	303	77	44	766	777							
2002	17'419	8'522	1'125	695	282	434	244	157	70								
2003	16'665	8'705	1'539	702	118	132	1'969	14									
2004	15'471	8'274	1'372	1'261	593	425	84										
2005	15'103	8'290	3'416	882	370	1'122						to be **predicted**					
2006	14'540	8'102	929	556	83												
2007	14'590	7'746	1'104	589													
2008	13'967	7'548	1'088														
2009	12'930	7'181															
2010	12'539																

- $X_{i,j}$ denote the payments for accident year i in development year j, i.e. $X_{i,j}$ is paid in accounting year $k = i + j$.

- Observed payments $\mathcal{D}_I = \{X_{i,j};\ i + j \leq I\}$ at time $I = 2010$.

Claims prediction and claims reserves

a.y.	development year j																
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1994	13'109	7'246	982	706	358	257	339	161	334	172	35	205	56	32	2	7	1
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- **Predict** the cash flows in the lower triangle

$$\mathcal{D}_I^c = \{X_{i,j};\ i + j > I\}\,.$$

(Nominal) best-estimate reserves

- Model the payments $X_{i,j}$ within a stochastic framework.
- The nominal best-estimate reserves at time I for the outstanding loss liabilities $X_{i,j} \in \mathcal{D}_I^c$ (lower triangle) are defined by

$$\mathcal{R}_I = \sum_{i+j>I} \mathbb{E}\left[X_{i,j} \middle| \mathcal{D}_I \right].$$

  This is the probability-weighted average of all future cash flows based on the latest information available (see Solvency II guidelines).

- Predictors $\mathbb{E}\left[X_{i,j} \middle| \mathcal{D}_I \right]$ have minimal prediction variance (optimal).
- For (stochastic) discounting we refer to W.-Merz [4].

Bayesian CL model of Hertig [1]

Define cumulative payments $C_{i,j} = \sum_{l=0}^{j} X_{i,l}$.

Model assumptions.

- Conditionally, given $\Theta = (\Theta_0, \ldots, \Theta_J)$,
  - ▷ $(C_{i,j})_{j=0,\ldots,J}$ are independent in i,
  - ▷ $(C_{i,j})_{j=0,\ldots,J}$ are Markov processes in j,
  - ▷ for $j = 0, \ldots, J$

$$\xi_{i,j} \overset{\text{def.}}{=} \log\left(\frac{C_{i,j}}{C_{i,j-1}}\right) \overset{(\mathrm{d})}{\sim} \mathcal{N}\left(\Theta_j, \sigma_j^2\right).$$

- The parameter has prior distribution $\Theta \overset{(\mathrm{d})}{\sim} \mathcal{N}(\mu, T)$.

Ultimate claim prediction

a.y. i	development year j 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\widehat{C}_{i,J}$
1994	13'109	7'246	982	706	358	257	339	161	334	172	35	205	56	32	2	7	1	
1995	14'457	7'581	589	487	124	74	128	50	474	12	72	63	141	286	2	10		
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2004	15'471	8'274	1'372	1'261	593	425	84					$\longrightarrow$						
2005	15'103	8'290	3'416	882	370	1'122						$\longrightarrow$						$C_{i,I-i}\prod_j f_j$
2006	14'540	8'102	929	556	83							$\longrightarrow$						
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- The expected ultimate claim (predictor) in this CL model is given by

$$\widehat{C}_{i,J} = \mathbb{E}\left[C_{i,J}\middle|\mathcal{D}_I\right] = C_{i,I-i}\prod_{j=I-i+1}^{J}\exp\left\{\Theta_j+\sigma_j^2/2\right\} = C_{i,I-i}\prod_{j=I-i+1}^{J} f_j.$$

Discussion of the model assumptions

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- CL factors $f_j = \exp\left\{\Theta_j + \sigma_j^2/2\right\}$ are **not known** $\Rightarrow$ Bayesian model.
- Payments in different accident years i are **not independent** $\Rightarrow$ multivariate model.
- Different portfolios $X_{i,j,n}$ for $n = 1, \ldots, N$ $\Rightarrow$ multivariate model.

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- Different portfolios $X_{i,j,n}$ for $n = 1, \ldots, N$ $\Rightarrow$ multivariate model.

Definition of the multivariate model

Define cumulative payments $C_{i,j,n} = \sum_{l=0}^{j} X_{i,l,n}$ of portfolio $n = 1, \ldots, N$ with corresponding individual log-link ratios

$$\xi_{i,j,n} = \log\left(\frac{C_{i,j,n}}{C_{i,j-1,n}}\right).$$

For $i \in \{1, \ldots, I\}$ and $j \in \{0, \ldots, J\}$ we define the random vectors

$$\begin{aligned} \xi_{i,j} &= (\xi_{i,j,1}, \ldots, \xi_{i,j,N})' \in \mathbb{R}^N, \\ \boldsymbol{\xi}_i &= (\xi_{i,0}', \ldots, \xi_{i,J}')' \in \mathbb{R}^a, \\ \boldsymbol{\xi} &= (\boldsymbol{\xi}_1', \ldots, \boldsymbol{\xi}_I')' \in \mathbb{R}^d, \end{aligned}$$

with $a = N(J+1)$ and $d = aI$.

Bayesian multivariate CL model

Model assumptions.

- Conditionally, given $\boldsymbol{\Theta} \in \mathbb{R}^a$,
  - ▷ $\boldsymbol{\xi}|_{\{\boldsymbol{\Theta}\}}$ is **multivariate** Gauss with covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$,
  - ▷ and mean $\mathbb{E}\left[\boldsymbol{\xi}_i \middle| \boldsymbol{\Theta}\right] = \boldsymbol{\Theta}$ for all $i \in \{1, \ldots, I\}$.
- The parameter has multivariate **prior** distribution $\boldsymbol{\Theta} \overset{(\mathrm{d})}{\sim} \mathcal{N}(\boldsymbol{\mu}, T)$.

This implies conditionally, given $\boldsymbol{\Theta} \in \mathbb{R}^a$,

$$\boldsymbol{\xi}|_{\{\boldsymbol{\Theta}\}} \overset{(\mathrm{d})}{\sim} \mathcal{N}(A\boldsymbol{\Theta}, \Sigma),$$

for an appropriate matrix $A = (\mathbb{1}, \ldots, \mathbb{1})' \in \mathbb{R}^{d \times a}$.

Consequences of the model assumptions

- Appropriate choices of Σ allow to model **any correlation structure**, such as inflation, within the triangles and between the triangles:

$$\boldsymbol{\xi}|_{\{\boldsymbol{\Theta}\}} \stackrel{\text{(d)}}{\sim} \mathcal{N}(A\boldsymbol{\Theta}, \Sigma).$$

- The choice of the prior distribution allows to implement **parameter uncertainty** and **expert knowledge** about CL factors

$$\boldsymbol{\Theta} \stackrel{\text{(d)}}{\sim} \mathcal{N}(\boldsymbol{\mu}, T).$$

- Thus, we obtain a classical **Bayesian multivariate model**.

Theorems (1/2)

Theorem 1 (unconditional distribution). We have

$$\boldsymbol{\xi} \overset{(\mathrm{d})}{\sim} \mathcal{N}(A\boldsymbol{\mu},\ S = \Sigma + ATA').$$

$\square$

Assume we have observations $\boldsymbol{\xi}^{\mathcal{D}_I}$. Consider the decomposition

$$\boldsymbol{\xi} \ \mapsto \ \left(\boldsymbol{\xi}^{\mathcal{D}_I}, \boldsymbol{\xi}^{\mathcal{D}_I^c}\right) = \left(P^{\mathcal{D}_I}\boldsymbol{\xi}, P^{\mathcal{D}_I^c}\boldsymbol{\xi}\right),$$

where $P_{\mathcal{D}_I}$ and $P_{\mathcal{D}_I^c}$ are the appropriate projections on the triangles.

Theorems (2/2)

Theorem 2 (predictive distribution). We have

$$\boldsymbol{\xi}^{\mathcal{D}_I^c}|_{\{\boldsymbol{\xi}^{\mathcal{D}_I}\}} \stackrel{(\mathrm{d})}{\sim} \mathcal{N}(\boldsymbol{\mu}^{\mathrm{post}}, S^{\mathrm{post}}),$$

with predictive mean (credibility formula)

$$\boldsymbol{\mu}^{\mathrm{post}} = P^{\mathcal{D}_I^c}(A\boldsymbol{\mu}) + S_{\mathcal{D}_I^c,\mathcal{D}_I}\ \left(S_{\mathcal{D}_I}\right)^{-1}\left(\boldsymbol{\xi}^{\mathcal{D}_I} - P^{\mathcal{D}_I}(A\boldsymbol{\mu})\right),$$

and predictive covariance matrix

$$S^{\mathrm{post}} = S_{\mathcal{D}_I^c} - S_{\mathcal{D}_I^c,\mathcal{D}_I}\ \left(S_{\mathcal{D}_I}\right)^{-1} S_{\mathcal{D}_I,\mathcal{D}_I^c}.$$

□

Interpretation and results (1/2)

- We can choose

  ⋆ **any correlation structure** Σ between the $\xi_{i,j,n}$'s and
  ⋆ **any prior information** $\boldsymbol{\mu}$ and T on the parameter space $\boldsymbol{\Theta}$,

  and then Theorem 2 provides the **posterior distribution** of $\boldsymbol{\xi}^{\mathcal{D}_I^c}$, given the observations $\boldsymbol{\xi}^{\mathcal{D}_I}$.

- The ultimate claim predictor is given by

$$\widehat{C}_{i,J,n} = \mathbb{E}\left[C_{i,J,n} \middle| \mathcal{D}_I\right] = C_{i,I-i,n}\ \exp\left\{\mathbf{e}'_{i,n}\ \boldsymbol{\mu}^{\mathrm{post}} + \frac{1}{2}\ \mathbf{e}'_{i,n}\ S^{\mathrm{post}}\ \mathbf{e}_{i,n}\right\},$$

  for appropriate projections $\mathbf{e}_{i,n}$.

Interpretation and results (2/2)

- Conditional mean square error of prediction (MSEP)

$$\mathrm{msep}_{\sum_{i,n} C_{i,J,n}|\mathcal{D}_I}\left(\sum_{i,n} \widehat{C}_{i,J,n}\right) = \sum_{i,l,n,m} \widehat{C}_{i,J,n}\widehat{C}_{l,J,m}\Big(\exp\left\{\mathbf{e}_{i,n}' \, S^{\mathrm{post}} \, \mathbf{e}_{l,m}\right\} - 1\Big).$$

- A similar formula is available for the one-year uncertainty, called claims development result (CDR), see Merz et al. [2].

- We present a case study with $N = 2$ portfolios:

  ⋆ general liability and motor liability portfolios,

  ⋆ fixed correlation $\rho \in [0, 1)$ along accounting years (diagonals).

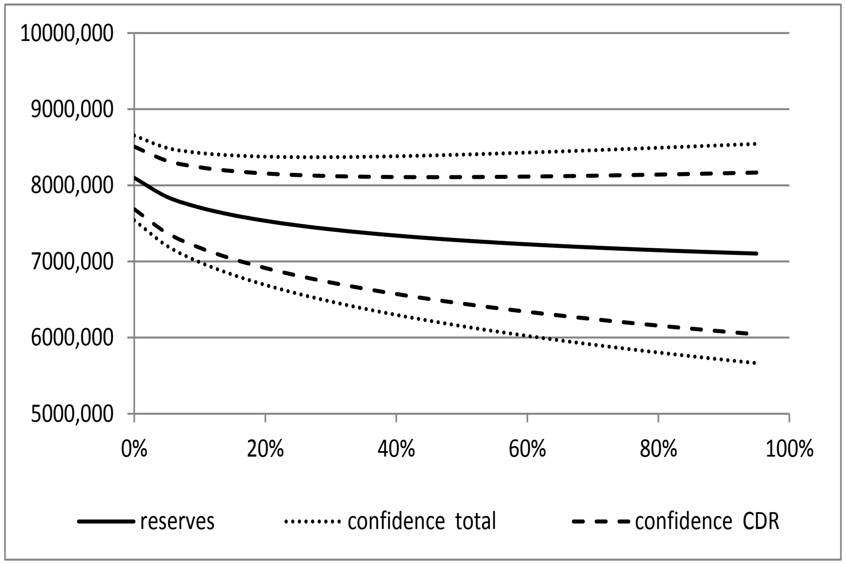
  For more details see Merz et al. [2].

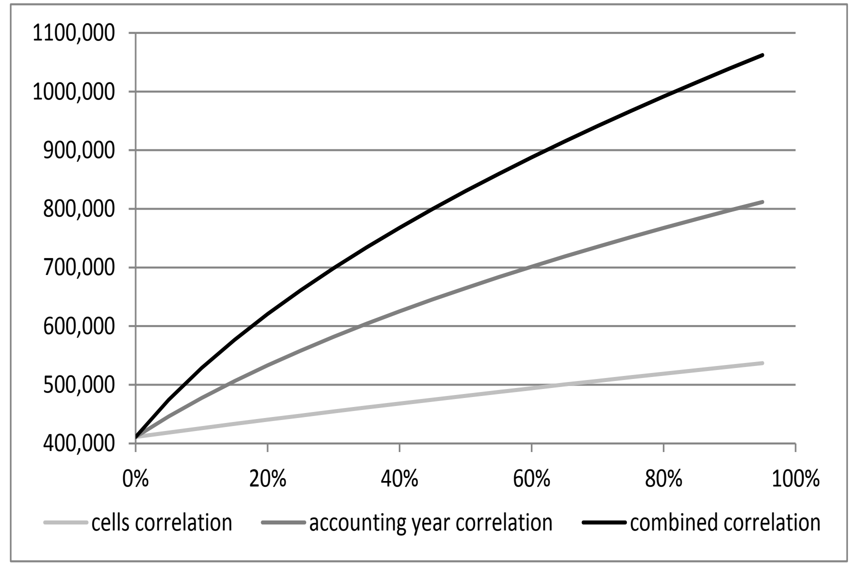
Case study from Merz et al. [2]

ρ	claims reserves	total uncertainty $\mathrm{msep}^{1/2}$ absolute	total uncertainty $\mathrm{msep}^{1/2}$ in % reserves	CDR uncertainty $\mathrm{msep}^{1/2}$ absolute	CDR uncertainty $\mathrm{msep}^{1/2}$ in % reserves
0%	8097,585	556,337	6.9%	410,964	5.1%
10%	7707,647	718,021	9.3%	528,494	6.9%
20%	7533,921	843,029	11.2%	620,545	8.2%
30%	7421,690	948,802	12.8%	698,663	9.4%
40%	7340,040	1042,186	14.2%	767,687	10.5%
50%	7276,834	1126,728	15.5%	830,183	11.4%
60%	7225,968	1204,555	16.7%	887,706	12.3%
70%	7183,917	1277,057	17.8%	941,278	13.1%
80%	7148,458	1345,205	18.8%	991,616	13.9%
90%	7118,096	1409,706	19.8%	1039,243	14.6%

$\Longrightarrow$ We observe a substantial increase in prediction uncertainty.

Case study from Merz et al. [2]





(a) lhs: claims reserves and 1-std.dev. confidence interval for total uncertainty and CDR uncertainty as a function of $\rho \in [0, 1)$.

(b) rhs: comparison of models (i) accounting year correlation, (ii) point-wise cells correlation and (iii) combined correlation for CDR uncertainty and $\rho \in [0, 1)$.

Summary

- We present a Bayesian multivariate reserving model that allows for
  - ⋆ **any correlation structure** between data, and
  - ⋆ for modeling **parameter uncertainty**.

- We have closed form solutions for
  - ⋆ the predictive distribution of $\boldsymbol{\xi}^{\mathcal{D}_I^c}$, given $\boldsymbol{\xi}^{\mathcal{D}_I}$,
  - ⋆ the ultimate claim predictor and the claims reserves,
  - ⋆ MSEP for total uncertainty and one-year uncertainty.

- Our model allows for a

  bottom-up calibration of correlation

  in solvency models.

References

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