Linear PDEs perturbed by Gaussian Noise

Bohdan Maslowski

Hannover, January 2015

General Case

Let (B_t) be a continuous centered Gaussian process on a complete probability space with sigma-algebra generated by the process, suppose that its covariance $R(t,s),\ s,t \in [0,T]$ may be expressed as

$$R(t,s) = \int_0^{min(s,t)} K(t,r)K(s,r)dr,$$

where K is square integrable and

$$\sup_{t\in[0,T]} \int_0^t K(t,s)^2 ds < \infty.$$

Furthermore, assume that there exists a Wiener process (W_t) such that

$$B_t = \int_0^t K(t,s)dW_s, \quad t \in [0,T].$$

General Case

Assume further

(K1) For all $s \in (0, T]$, $K(\cdot, s)$ has a bounded variation on $(s, T]$ and

$$\int_0^T |\mathcal{K}|((s, T], s)^2 ds < \infty.$$

Set

$$(K^* \varphi)(s) = \varphi(s) K(T, s) + \int_s^T (\varphi(t) - \varphi(s)) \mathcal{K}(\mathrm{d}t, s). \qquad (1)$$

for $\varphi \in \mathcal{E}$, the space of V-valued deterministic step functions.

General case

For $x, y \in V$ define

$$\langle x 1_{[0,t]}, y 1_{[0,s]} \rangle_{\mathcal{H}} := \langle x, y \rangle_V R(t,s), \quad (t,s) \in [0,T]^2. \tag{2}$$

The inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ can be extended (by linearity) to $\mathcal{E}$ and $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ forms a pre-Hilbert space. The completion of $\mathcal{E}$ with respect to the above scalar product is denoted by $\mathcal{H}$. Stochastic integral w.r.t. β is defined in the standard way on $\mathcal{E}$ and extended to $\mathcal{H}$. We have

$$\|\varphi\|_{\mathcal{H}}^2 = \|K^* \varphi\|_{L^2([0,T],V)}^2.$$

and

$$\beta(\varphi) = \int_0^T (K^* \varphi)(t) \mathrm{d} W_t, \quad \mathbb{P} - \text{a.s.}. \tag{3}$$

General case

(K2) For some $\alpha \in \left(0, \frac{1}{2}\right)$, the kernel K satisfies the following:

- For all $s \in (0, T)$ the function $K(\cdot, s) : (s, T] \to \mathbb{R}$ is differentiable in the interval (s, T) and both $K(t, s)$ and the derivatives $\frac{\partial K}{\partial t}(t, s)$ are continuous at every $t \in (s, T)$.
- There exist a konstant $c > 0$ such that

$$\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t - s)^{\alpha - 1} \left(\frac{s}{t} \right)^{-\alpha}, \qquad (4)$$

$$\int_s^t K(t, r)^2 \mathrm{d}r \leq c(t - s)^{2\alpha + 1}$$

for $0 \leq s < t \leq T$.

General case

Theorem

Let $(K1)$ be satisfied. Consider the seminorm

$$\|\varphi\|_{\mathcal{H}_R}^2 := \int_0^T |\varphi(s)|_V^2 K(s^+, s)^2 \mathrm{d}s + \int_0^T \left(\int_s^T |\varphi(t)|_V |\mathcal{K}|(\mathrm{d}t, s) \right)^2 \mathrm{d}s, \tag{5}$$

defined on $\mathcal{E}$. Denote by $\mathcal{H}_R$ the completion of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathcal{H}_R}$. Then $\mathcal{H}_R$ is continuously embedded in $\mathcal{H}$.

1. *There exists a finite constant $c_1 > 0$ such that*

$$\|\varphi\|_{\mathcal{H}} \leq c_1 \|\varphi\|_{b\mathcal{B}([0,T];V)}$$

for all $\varphi \in b\mathcal{B}([0,T];V)$.

General case

Theorem

Suppose further (for simplicity)that $K(s^+, s) = 0$ *for all* $0 < s < T$. *If* $(K2)$ *is satisfied then there exists a finite constant* $c_3(\alpha) > 0$ *such that*

$$\|\varphi\|_{\mathcal{H}} \leq c_3(\alpha)\|\varphi\|_{L^{\frac{2}{1+2\alpha}}([0,T];V)}$$

for each $\varphi \in L^{\frac{2}{1+2\alpha}}([0, T]; V)$.

Cylindrical Process

Let (K1) be satisfied. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, U be a real separable Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle_U$ and $T > 0$. Given an ONB basis (e_n) of the space U we define the cylindrical Gaussian Volterra process (with covariance kernel K) as a formal sum

$$B_t = \sum_{n=1}^{\infty} \beta_n(t) e_n,$$

where $(\beta_n(t))$ is a sequence of pairwise independent one-dimensional Gaussian Volterra processes with the same covariance kernel. The series does not converge in the space U but may be understood as usual as a family of random linear functionals (or may be shown to be convergent in any Hilbert space U_1 such that the embedding $U \hookrightarrow U_1$ is Hilbert-Schmidt).

Stochastic integral

Let $G : [0, T] \to \mathcal{L}(U, V)$ be an operator-valued function such that $G(\cdot)e_n \in \mathcal{H}$ for $n \in \mathbf{N}$, and B be a standard cylindrical Gaussian Volterra process in U.
Define

$$\int_0^T G\, dB^H := \sum_{n=1}^{\infty} \int_0^T G e_n \, d\beta_n$$

provided the infinite series converges in $L^2(\Omega, V)$.

Linear equations

$$\left\{ \begin{array}{rcll} \mathrm{d}X_t & = & AX_t \, \mathrm{d}t + \Phi \, \mathrm{d}B_t, & t \geq 0 \\ X_0 & = & x, & \mathbb{P} - \mathrm{a.s.} \end{array} \right. \quad (6)$$

where $A : \mathrm{Dom}(A) \to V$, $\mathrm{Dom}(A) \subset V$, an infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on V, $\Phi \in \mathcal{L}(U, V)$ and $x \in V$.

$$X_t = S(t)x + \int_0^t S(t-s)\Phi \mathrm{d}B_s =: S(t)x + Z(t), \quad \mathbb{P} - \mathrm{a.s.} \qquad (7)$$

for $t \geq 0$.

Linear equations

- (A1) *For all* $T > 0$, K *satisfies (K1) on* $[0, T]$ *and induces a non-atomic measure* $\mathcal{K}$. *Moreover,* $\Phi \in \mathcal{L}_2(U, V)$
- (A2) *For all* $T > 0$, K *satisfies (K2) on* $[0, T]$ *and for all* $s \in (0, T]$, $S(s)\Phi$ *is a Hilbert-Schmidt operator such that*

$$\|S(\cdot)\Phi\|_{\mathcal{L}_2(U,V)} \in L^{\frac{2}{1+2\alpha}}(0, T). \tag{8}$$

Linear equations

Proposition

If at least one of the conditions (A1) and (A2) holds, then the process $Z = (Z_t, t \geq 0)$, *is well defined* V*-valued Gaussian process and its sample paths are* $\mathbb{P}$*-almost surely in* $L^2([0, T]; V)$ *for all* $T > 0$.

Linear equations

Proposition

Assume that for all $T > 0$*,* K *satisfies (K2) on* $[0, T]$ *and for all* $s \in [0, T]$*,* $S(s)\Phi$ *is a Hilbert-Schmidt operator such that*

$$t \to t^{-\beta} \|S(t)\Phi\|_{\mathcal{L}_2(U,V)} \in L^{\frac{2}{1+2\alpha}}(0, T). \tag{9}$$

for some $\beta > 0$*. Then the process* Z *has a Hölder continuous version in* V*.*

Linear equations

Corollary (Sufficient condition for (A2))

If for all $T > 0$ *there exist finite constants* $c > 0$ *and* $0 \leq \gamma < \frac{1}{2} + \alpha$ *such that*

$$\|S(t)\Phi\|_{\mathcal{L}_2(U,V)} \leq ct^{-\gamma}, \quad t \in (0, T]$$

then there exists a Hölder continuous version of the process Z *in* V*.*

FBM

Definition

Let H be an element of $(0,1)$ (the Hurst parameter). A continuous centered Gaussian process $\beta^H(t)$, $t \in \mathbb{R}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called fractional Brownian motion if

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R}. \tag{10}$$

FBM

Let $K_H(t,s)$ for $0 \le s \le t \le T$ be the kernel function

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\tfrac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\tfrac{s}{u}\right)^{\frac{1}{2}-H}\right)du$$

where $c_H = \left[\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)}\right]^{\frac{1}{2}}$ and $\Gamma(\cdot)$ is the gamma function.
The operator $\mathcal{K}_H^*$ is given by

$$\mathcal{K}_H^*\varphi(t) := \varphi(t)K_H(T,t) + \int_t^T (\varphi(s)-\varphi(t))\frac{\partial K_H}{\partial s}(s,t)ds$$

for $\varphi \in \mathcal{E}$.

Example (Parabolic)

Consider the initial boundary value problem for stochastic parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= Lu(t,x) + \xi(t,x), \quad (t,x) \in \mathbf{R}_+ \times D, \\ u(0,x) &= u_0(x), \quad x \in D, \\ u(t,x) &= 0, \quad t \in \mathbf{R}_+, \, x \in \partial D, \end{aligned} \tag{11}$$

where $D \subset \mathbf{R}^d$ is a bounded domain with a smooth boundary, L is a second order uniformly elliptic operator on D and η is a noise process that is the formal time derivative of a space dependent fractional Brownian motion.

- rewrite the parabolic system as an infinite dimensional stochastic differential equation:

$U = L^2(D)$, $V = L^2(D)$, $\Phi = Id$; we get (A1) with $\rho = d/4$, so $Z \in \mathcal{C}^{\beta}([0,T], D_A^{\delta})$ for $\delta + \beta + \frac{d}{4} < H$.

Boundary and Pointwise Noise

$$\frac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi), \quad (t,\xi) \in D \subset \mathbb{R}^n,$$
$$u(0,\xi) = x(\xi),$$
$$\frac{\partial u}{\partial \nu}(t,\xi) = \eta^H(t,\xi), \quad (t,\xi) \in \partial D$$

(Neumann type boundary noise), or

$$u(t,\xi) = \eta^H(t,\xi), \quad (t,\xi) \in \partial D$$

(Dirichlet type boundary noise).

Boundary and Pointwise Noise

Modelled as

$$Z^x(t) = S(t)x + \int_0^t S(t-r)\Phi dB^H(r), \quad t \geq 0,$$

where $\Phi = (A - \hat{\beta} I)N$, N is the Neumann (or Dirichlet) map, the state space is $V = L^2(D)$, and B^H is a cylindrical fBm on a separable Hilbert space $U \subset L^2(\partial D)$.

Conditions for existence and time Hölder continuity of the solution :

- $d = 1$: $\frac{1}{4} < H$ (Neumann) and $\frac{3}{4} < H$ (Dirichlet).
- $d \geq 2$: $\frac{1}{2} + \frac{1}{4}(d-1) < H$ (Neumann).

Boundary and Pointwise Noise

$$\frac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi) + \delta_z \eta_t^H, \quad (t,\xi) \in D$$

$$u(0,\xi) = x(\xi),$$

$$\frac{\partial u}{\partial \nu}(t,\xi) = 0, \quad (t,\xi) \in \partial D$$

(pointwise noise, δ_z - Dirac distribution at $z \in D$).

Modelled as

$$Z^x(t) = S(t)x + \int_0^t S(t-r)\Phi d\beta^H(r), \quad t \geq 0,$$

in $V = L^2(D)$, where Φ is a distribution, i.e. $\Phi \in (D_A^\delta)^*$ for $\delta > \frac{d}{4}$. We have a (Hölder) continuous solution for $\delta < H$, i.e. for $\frac{d}{4} < H$.

Equations with Multiplicative Noise

Consider the equation with finite-dimensional (fBm)

$$dX(t) = A(t)X(t)dt + \sum_{k=1}^{m} B_k X(t) d\beta_k^H(t) \tag{12}$$

$$X(0) = x_0$$

where $(A(t))$ generates a strongly continuous family of operators $(U_0(t,s)),\ t \geq s$,

$$\frac{\partial}{\partial s} U_0(t,s) = -U_0(t,s)A(s) \tag{13}$$

$$\frac{\partial}{\partial t} U_0(t,s) = A(t)U_0(t,s) \tag{14}$$

Equations with Multiplicative Noise

(H1) The family of closed operators $(A(t),\ t \in [0, T])$ defined on a common domain $D := \mathrm{Dom}(A(t))$ for $t \in [0, T]$ generates a strongly continuous evolution operator $(U_0(t, s),\ 0 \leq s \leq t \leq T)$ on V.

(H2) The collection of linear operators $(B_1, \dots, B_m)$ generate mutually commuting strongly continuous groups $(S_1(s), \dots, S_m(s), s \in \mathbb{R})$ wich commute with $A(t)$ on D for each $t \in [0, T]$. For $i, j \in \{1, \dots, m\}$, $\mathrm{Dom}(B_i B_j) \supset D$, $\mathrm{Dom}(A^*(t)) = D^*$ is independent of t and $D^* \subset \bigcap_{i,j=1}^{m} \mathrm{Dom}(B_i^* B_j^*)$ where $*$ denotes the topological adjoint.

(H3) The family of linear operators $(\tilde{A}(t),\ t \in [0, T])$ where $\tilde{A}(t) = A(t) - Ht^{2H-1} \sum_{j=1}^{m} B_j^2$, $\mathrm{Dom}(\tilde{A}(t)) = D$ for each $t \in [0, T]$, generates a strongly continuous evolution operator on V, $(U(t, s), 0 \leq s \leq t \leq T)$.

Equations with Multiplicative Noise

A $\mathcal{B}([0, T]) \otimes \mathcal{F}$ measurable stochastic process $(X(t),\ t \in [0, T])$ is said to be

(i) a *strong solution* of (12) if $X(t) \in D$ a.s. $\mathbb{P}$ and

$$X(t) = x_0 + \int_0^t A(s)X(s)ds + \sum_{j=1}^{m} \int_0^t B_j X(s) d\beta_j^H(s) \qquad \text{a.s.} \quad (15)$$

for $t \in [0, T]$.

(ii) a *weak solution* of (12) if for each $z \in D^*$

$$< X(t), z > = < x_0, z > + \int_0^t < X(s), A^*(s)z > ds \qquad (16)$$

$$+ \sum_{j=1}^{m} \int_0^t < X(s), B_j^* z > d\beta_j^H(s) \qquad \text{a.s.} \qquad (17)$$

for $t \in [0, T]$ and

Equations with Multiplicative Noise

(iii) a *mild solution* of (12) if

$$X(t) = U_0(t,0)x_0 + \sum_{j=1}^{m} \int_0^t U_0(t,s)B_j X(s)d\beta_j^H(s) \qquad \text{a.s.} \qquad (18)$$

for $t \in [0, T]$,

where the stochastic integrals in (15)–(18) are defined in the Skorokhod sense.

Equations with Multiplicative Noise

Theorem

Assume that $H > \frac{1}{2}$ and (H1)–(H3) are satisfied. There is a weak solution of (12). If $x_0 \in D$, then there is a strong solution of (12). If $B_j \in \mathcal{L}(V)$ for $j \in \{1, \dots, m\}$, then there is a mild solution of (12) which is unique in the space $\mathrm{Dom}\delta_H \cap L^2(\Omega; \tilde{\mathcal{H}})$, where δ_H denotes the divergence operator based on β^H. In each case the solution $(X(t),\ t \in [0, T])$ is given as follows

$$X(t) = \prod_{j=1}^{m} S_j(\beta_j^H(t))U(t,0)x_0 \tag{19}$$

for $t \in [0, T]$.

For $H < \frac{1}{2}$ there exists a weak solution given by formula (19) in the "'parabolic"' case (by approximations, using Cheredito-Nulart result on closedness of the extension of Skorokhod integral operator).

Equations with Multiplicative Noise - Existence and Uniqueness

Proof: Existence in the "'strong"' case: By fractional Ito formula, the other cases by approximations of the initial value (Malliavin derivatives in the Ito formula may be easily calculated).

Uniqueness (for simplicity, from now on $m = 1$, $\beta_1^H =: \beta^H$, $B_j =: B$, $S_1 =: S$).

$$X_t = U_0(t,0)x + \int_0^t U_0(t,r)BX_r d\beta_r^H,$$
$$Y_t = U_0(t,0)x + \int_0^t U_0(t,r)BY_r d\beta_r^H,$$

Define the process $Z = \{Z_t, t \in [0,T]\}$ as

$$Z_t = X_t - Y_t, \;\; t \in [0,T].$$

Equations with Multiplicative Noise - Existence and Uniqueness

Let

$$X_t = \sum_{n=0}^{+\infty} X_n(t), \;\; Y_t = \sum_{n=0}^{+\infty} Y_n(t), \;\; t \in [0, T],$$

be the respective Wiener chaos decompositions. Show (by induction) $Z_n = X_n - Y_n = 0$. We have $Z_0 = 0$ hence

$$\sum_{n=1}^{+\infty} Z_n(t) = \sum_{n=0}^{+\infty} \int_0^t U_0(t,s) B Z_n(s) d\beta_s^H.$$

Since $Z_0 \in \mathcal{H}_0$ then

$$\mathcal{H}_1 \ni \int_0^t U_0(t,s) B Z_0(s) d\beta_s^H = 0, \;\; t \in [0, T],$$

and consequently

$$Z_1(t) = \int_0^t U_0(t,s) B Z_0(s) d\beta_s^H = 0$$

for any $t \in [0, T]$ because $Z_1 \in \mathcal{H}_1$

Equations with Multiplicative Noise - Existence and Uniqueness

Suppose $Z_n = 0$ for some fixed $n \in \mathbf{N}$. By commutativity

$$\int_0^t U_0(t,s) B Z_n(s) d\beta_s^H = \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_1} U_0(t,s) B^n Z_0(s) d\beta_s^H d\beta_{t_1}^H \dots d\beta_{t_{n-1}}^H$$

is zero for any $t \in [0, T]$ and the expression belongs to $\mathcal{H}_{n+1}$. Moreover, $Z_{n+1} \in \mathcal{H}_{n+1}$ thus

$$Z_{n+1}(t) = \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_1} U_0(t,s) B^n Z_0(s) d\beta_s^H d\beta_{t_1}^H \dots d\beta_{t_{n-1}}^H = 0$$

for $t \in [0, T]$.

Examples

Let

$$\begin{array}{rcl} dX_t & = & AX_t dt + bX_t d\beta_t^H, \ \ t > 0, \\ X_0 & = & x, \end{array} \tag{20}$$

where $A : \mathrm{Dom}(A) \subset V \to V$ is the generator of a strongly continuous semigroup $\{S_A(t), t \geq 0\}$ and $b \in \mathbf{R} \setminus \{0\}$. Then

$$X_t = \exp\left\{b\beta_t^H - \frac{1}{2}b^2 t^{2H}\right\} S_A(t)x, \ \ 0 \leq s \leq t < +\infty,$$

and since there exist some constants $M > 0, \omega \in \mathbf{R}$ such that

$$\|S_A(t)\|_{\mathcal{L}(V)} \leq M\mathrm{e}^{\omega t}, \ \ t \geq 0,$$

we have that

$$|X_t|_V \leq M \exp\left\{b\beta_t^H - \frac{1}{2}b^2 t^{2H} + \omega t\right\}|x|_V \to 0 \tag{21}$$

a.s. as $t \to \infty$ (the solution is pathwise stabilized by noise)

Examples

However, for any $p > 0$, taking for simplicity $V = \mathbf{R}$, $A = \omega$, $x \neq 0$

$$\mathbb{E}|X_t|^p = |x|^p \exp\left\{p\omega t - \frac{1}{2} b^2 p t^{2H} + pbB_t^H\right\}, \ t \geq 0, \ p > 1,$$

hence for each $\epsilon > 0$ there exists $\tilde{C}_\epsilon > 0$ such that

$$\mathbb{E}\left[|X_t|_V^p\right] = |x|^p \exp\left\{\hat{c} t^{2H} + p\omega t\right\} \geq \tilde{C}_\epsilon \exp\{(\hat{c} - \epsilon) t^{2H}\}, \ t \geq 0,$$

where $\hat{c} = \frac{1}{2} b^2 (p^2 - p)$, so for $p > 1$ the p-th moment of the solution is destabilized by noise.

Examples

$$\frac{\partial u}{\partial t}(t,\xi) = L(t,\xi)u(t,\xi) + b\frac{d\beta^H}{dt}u \tag{22}$$
$$u(0,\xi) = x_0(\xi)$$

for $(t,\xi) \in [0,T] \times \mathcal{O}$

$$\left(\frac{\partial u}{\partial \xi}\right)^{\alpha}(t,\xi) = 0, \quad (t,\xi) \in [0,T] \times \partial\mathcal{O}, \ \alpha \in \{1,\ldots,k-1\}$$

where $k \in \mathbb{N}$, $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class C^k, $b \in \mathbb{R} \setminus \{0\}$ and

$$L(t,\xi) := \sum_{|\alpha| \le 2k} a_\alpha(t,\xi)D^\alpha \tag{23}$$

is a strongly elliptic operator on $\mathcal{O}$, uniformly in $(t,\xi) \in [0,T] \times \overline{\mathcal{O}}$ and $a_\alpha(t,\cdot) \in C^{2k}(\overline{\mathcal{O}})$ for each $t \in [0,T]$.

Examples

The equation (22) is rewritten in the form

$$
\begin{aligned}
dX(t) &= A(t)X(t)dt + BX(t)d\beta^H(t) \\
X(0) &= x_0 \in V
\end{aligned}
\tag{24}
$$

for $t \in [0, T]$, where $V = L^2(\mathcal{O})$, $(A(t)u)(\xi) = L(t,\xi)u(t,\xi)$, $\mathrm{Dom}(A(t)) = D = H^{2k}(\mathcal{O}) \cap H_0^k(\mathcal{O})$ and $B = bI \in \mathcal{L}(V)$. It is assumed that

$$
\sup_{\xi \in \mathcal{O}} |a_\alpha(t,\xi) - a_\alpha(s,\xi)| \leq M|t-s|^\gamma \tag{25}
$$

Examples

$$\frac{\partial u}{\partial t}(t,\xi) = a\frac{\partial^2 u}{\partial \xi^2}(t,\xi) + b\frac{\partial u}{\partial \xi}(t,\xi)\frac{d\beta^H}{dt}(t) \tag{26}$$

$$[S(t)x_0](\xi) = x_0(\xi + bt) \tag{27}$$

The ellipticity condition (H3) is satisfied if $a > Ht^{2H-1}b^2$. The solution may be expressed

$$(S_\Delta x)(\xi) = \int_{\mathbb{R}} (4\pi t)^{-1/2} \exp\Big[-\frac{1}{4t}(\xi-\eta)^2\Big]^2 x(\eta)d\eta \tag{28}$$

$$X(t) = S(\beta^H(t))S_\Delta\Big(at - \frac{1}{2}b^2t^{2H}\Big)x_0. \tag{29}$$

So the problem is "well posed" for $0 \le t \le T$, where $T = \left(\frac{2a}{b^2}\right)^{1/(2H-1)}$.

Examples

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial \xi^4} - \alpha u + \frac{\partial u}{\partial \xi}\frac{d\beta^H(t)}{dt} \tag{30}$$

$$u(0,\xi) = x_0(\xi) = \sin\xi$$

$$\tilde{A}(t) = L - tH^{2H-1}B^2 = -\frac{\partial^4}{\partial \xi^4} - \alpha I - tH^{2H-1}\frac{\partial^2}{\partial \xi^2} \tag{31}$$

The solution has the form

$$X(t) = S(\beta^H(t))U(t,0)x_0. \tag{32}$$

Examples

Setting $[U(t,0)x_0](\xi) = \varphi(t)\sin\xi$ we obtain

$$\dot{\varphi}(t)\sin\xi = -\varphi(t)\sin\xi - \alpha\varphi(t)\sin\xi + Ht^{2H-1}\varphi(t)\sin\xi$$
$$\varphi(0) = 1.$$

and hence

$$X(t) = \sin\left(\xi + \beta^H(t)\right)\exp\left[-(1+\alpha)t + \frac{1}{2}t^{2H}\right]. \tag{33}$$

It follows that

$$\lim_{t\to\infty} |X(t)| = \infty, \qquad \text{a.s.}$$

so the noise destabilizes the equation.

General Case

Theorem

Assume (K1) and let $F \in C^{1,2}([0,T] \times R)$ has at most exponential growth in the second variable, uniform in t. Then $F(t, B_t)$ belongs to $\mathbb{D}^{1,2}$ and we have

$$F(t, B_t) = F(0,0) + \int_0^t D_t F(s, B_s) ds + \int_0^t D_x F(s, B_s) dB_s$$

$$+ \frac{1}{2} \int_0^t D_x^2 F(s, B_s) dR(s),$$

where $R(s) := R(s,s)$ (under (K1) R has bounded variation).

General Case

The natural candidate for the evolution system $U(t,s)$ would be the one corresponding to the equation

$$y(t) = y_0 + \int_0^t A(s)y(s)ds - \int_0^t B^2 y(s)dR(s), \quad t \in [0, T].$$

If we additionaly assume that $R \in C^1([0, T])$ all results stated above (in the regular case) remain true with t^{2H} replaced by $R(t)$ and Ht^{2H-1} by $R'(t)$.

Random Evolution System

Consider

$$\begin{aligned} \mathrm{d}Y_t &= AY_t dt + BY_t d\beta_t^H, \ t > s, \\ Y_s &= x, \end{aligned} \tag{34}$$

assume that $(\tilde{A}(t))$ generates the "'parabolic"' strongly evolution system $\{U(t,s), 0 \leq s \leq t \leq T\}$ on V.

$$\begin{aligned} &(U(t,s)(V) \subset D, \\ &\|U(t,s)\|_{\mathcal{L}(V)} \leq C_U, \\ &\left\|\frac{\partial}{\partial t} U(t,s)\right\|_{\mathcal{L}(V)} = \|\tilde{A}(t)U(t,s)\|_{\mathcal{L}(V)} \leq \frac{C_U}{t-s}, \\ &\|\tilde{A}(t)U(t,s)(\tilde{A}(s) - \bar{\omega}I)^{-1}\|_{\mathcal{L}(V)} \leq C_U \end{aligned} \tag{35}$$

for some constant $C_U > 0$ and any $0 \leq s < t \leq T$.

Random Evolution System

What is the random evolution system defined by the equation (34)? It may be verified that the equation has a weak solution $\{U_Y(t,s)x, s \leq t \leq T\}$ given by a formula

$$U_Y(t,s)x = S(B_t^H - B_s^H)U(t-s,0)x, \; s \leq t \leq T, \tag{36}$$

for any initial value $x \in V$. Note that $U_Y(t,s)$ is not the same as

$$\bar{U}_Y(t,s) = S(B_t^H - B_s^H)U(t,s).$$

In one-dimensional case, $A = a$, $B = b$ we have

$$\bar{U}_Y(t,s) = S(B_t^H - B_s^H)U(t,s) = \exp\left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2\left(t^{2H} - s^{2H}\right)\right\}, \tag{37}$$

while

$$U_Y(t,s) == \exp\left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2\left(t - s)^{2H}\right)\right\}, \; 0 \leq s \leq t \leq T. \tag{38}$$

$U_Y(t,s)$ does not posses the composition (cocycle) property (the equation does not define RDS) while $\bar{U}_Y(t,s)$ does.

Affine equation

Theorem

Let $F : [0, T] \times V \to V$ *be a measurable function satisfying*

$(\mathrm{i})_{\mathrm{F}}$ *there exists a function* $\bar{L} \in L^1([0, T])$ *such that*

$$\|F(t,x) - F(t,y)\|_V \leq \bar{L}(t)\|x - y\|_V, \; x, y \in V, \; t \in [0, T].$$

$(\mathrm{ii})_{\mathrm{F}}$ *for some function* $\bar{K} \in L^1([0, T])$

$$\|F(t,0)\|_V \leq \bar{K}(t), \; t \in [0, T].$$

Then the equation

$$y(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F\big(r, y(r)\big)\,dr \tag{39}$$

has a unique solution in the space $\mathcal{C}([0, T]; V)$ *for a.e.* $\omega \in \Omega$ *and any initial value* $x \in V$.

Affine equation

In the Wiener case $H = 1/2$ the solution to the equation (39) is the so-called mild solution to the equation

$$\begin{aligned} dX_t &= AX_t dt + F(t, X_t)dt + BX_t dW_t, \\ X_0 &= x \in V. \end{aligned}$$

and is known to coincide with the weak solution. What can we say in the general case?

Affine equation

Theorem

Let the assumptions of Theorem 4 hold and $\{X_t, t \in [0, T]\}$ *be the solution to the equation* mild.rce *such that there exists a constant* $C_X < +\infty$

$$\max\left\{ \sup_{t\in[0,T]} \mathbb{E}\|X_t\|_V^4, \sup_{t\in[0,T]} \sup_{v\in[0,T]} \mathbb{E}\|D_v^H X_t\|_V^4 \right\} \le C_X. \tag{40}$$

In addition, let F *be Fréchet differentiable with respect to the space variable for any time* $t \in [0, T]$*. Suppose that there exists a function* $C \in L^4([0, T])$ *such that*

$$\max\{\|F(t,x)\|_V, \|F_x'(t,x)\|\} \le C(t), \ \ t \in [0, T], \tag{41}$$

holds. Then $\{X_t, t \in [0, T]\}$ *is a solution to the integral equation*

Affine equation

Theorem

$$X_t = x + \int_0^t AX_r dr + \int_0^t F(r, X_r) dr + \int_0^t BX_r d\beta_r^H$$
$$+ \int_0^t \alpha_H \int_0^T \int_r^t |v - w|^{2H-2} BU_Y(v, r) F'_x(r, X_r) D_w^H X_r dv dw dr$$

in a weak sense, i.e. for any $y \in D^*$, $t \in [0, T]$,

$$\langle X_t, y \rangle_V = \langle x, y \rangle_V + \int_0^t \langle X_r, A^* y \rangle_V dr$$
$$+ \int_0^t \langle F(r, X_r), y \rangle_V dr + \int_0^t \langle X_r, B^* y \rangle_V d\beta_r^H$$
$$+ \int_0^t \alpha_H \int_0^T \int_r^t |v - w|^{2H-2} \Big\langle U_Y(v, r) F'_x(r, X_r) D_w^H X_r, B^* y \Big\rangle_V dv dw$$

Affine equation

Consider a one-dimensional equation

$$dX_t = aX_t dt + bX_t d\beta_t^H, \ X_0 = 1, \tag{42}$$

$a, b \in \mathbf{R}$ are nonzero constants. In the previous notation,

$$dX_t = F(t, X_t)dt + BX_t d\beta_t^H, \ X_0 = 1,$$

where $F(t, x) = ax$, $A = 0$ and $B = bI$. Recall that

$$\bar{U}_Y(t, s) = S(\beta_t^H - \beta_s^H)U(t, s) = \exp\left\{ b(\beta_t^H - \beta_s^H) - \frac{1}{2}b^2\left(t^{2H} - s^{2H}\right) \right\}.$$

Then

$$X_t = \bar{U}_Y(t, 0) + \int_0^t \bar{U}_Y(t, r)F(r, X_r)dr \tag{43}$$

Affine equation

Theorem

Let the assumptions of Theorem 4 be satisfied and $F : [0, T] \to V$ *be a measurable function independent of a space variable such that* $\|F\|_V \in L^2([0, T])$*. Then the solution* $\{X_t^M, t \in [0, T]\}$ *to the affine equation (39) obtained in Theorem 4 having the form*

$$X_t^M = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r)dr \tag{44}$$

is a weak solution to the equation

$$\begin{aligned} dX_t &= \big(AX_t + F(t)\big)dt + BX_t d\beta_t^H, \\ X_0 &= x \in V. \end{aligned} \tag{45}$$

Affine equation

Corollary

For each $p \geq 1$ there exists a constant $c_p > 0$ depending only on p such that

$$\mathbb{E}\left[\|X_t\|_V^p\right] \leq c_p M \exp\left\{\frac{(p^2-p)b^2}{2}t^{2H} + p\omega t\right\}\|x\|_V^p$$
$$+ M t^{p-1}\int_0^t \exp\left\{\frac{(p^2-p)b^2}{2}(t-s)^{2H}\right. \tag{46}$$
$$\left. + p\omega(t-s)\right\}\|F(s)\|_V^p ds, \;\; t \geq 0. \tag{47}$$

In particular, if $F(t) \equiv F$ does not depend on $t \geq 0$, for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\mathbb{E}\left[\|X_t\|_V^p\right] \leq C_\epsilon \exp\{(\hat{c}+\epsilon)t^{2H}\}, \;\; t \geq 0, \tag{48}$$

holds with $\hat{c} = 1/2b^2(p^2-p)$.

Some References

- T.E.Duncan, B.Maslowski and B.Pasik-Duncan, Fractional Brownian motion and stochastic equations in Hilbert spaces, *Stoch. Dyn.* 2(2002), 225-250
- B.Maslowski and B.Schmalfuss, Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion, *Stochastic Anal.Appl.*, 22 (2004), 1577-1609
- T.E.Duncan, B.Maslowski and B.Pasik-Duncan, Stochastic equations in Hilbert space with a multiplicative fractional Gaussian noise, *Stoch. Process. Appl.* 115 (2005), 1357-1383
- J. Bartek, M. Garido-Atienza and B. Maslowski, Stochastic porous media equation driven by fractional Brownian motion, *Stoch. Dyn.* 13 (2013), 1350010 (33 pp.)
- B. Maslowski and J. Šnupárková, Stochastic equations with multiplicative fractional Gaussian noise in Hilbert space, submitted

Cylindrical fractional Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space,
$U = (U, \langle \cdot, \cdot \rangle_U, |\cdot|_U)$ be a separable Hilbert space.
A cylindrical process $\langle B^H, \cdot \rangle : \Omega \times \mathbf{R} \times U \to \mathbf{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (0,1)$ if

1. For each $x \in U \setminus \{0\}$, $\frac{1}{|x|_U} \langle B^H(\cdot), x \rangle$ is a standard scalar fractional Brownian motion with Hurst parameter H.
2. For $\alpha, \beta \in \mathbf{R}$ and $x, y \in U$,

$$\left\langle B^H(t), \alpha x + \beta y \right\rangle = \alpha \left\langle B^H(t), x \right\rangle + \beta \left\langle B^H(t), y \right\rangle \quad \text{a.s. } \mathbb{P}.$$

- $\langle B^H(t), x \rangle$ has the interpretation of the evaluation of the functional $B^H(t)$ at x,
- For $H = \frac{1}{2}$ it is standard cylindrical Wiener process in U.

Cylindrical FBM

We can associate $(B^H(t), t \in \mathbf{R})$ with a standard cylindrical Wiener process $(W(t), t \in \mathbf{R})$ in U formally by $B^H(t) = \mathbb{K}_H \left(\dot{W}(t)\right)$. For $x \in U \setminus \{0\}$, let $\beta_x^H(t) = \langle B^H(t), x\rangle$. It is elementary to verify from (**??**) that there is a scalar Wiener process $(w_x(t), t \in \mathbf{R})$ such that

$$\beta_x^H(t) = \int_0^t K_H(t,s)\, dw_x(s) \tag{49}$$

for $t \in \mathbf{R}$.
Furthermore, if $V = \mathbf{R}$, then $w_x(t) = \beta_x^H \left((\mathcal{K}_H^*)^{-1} 1_{[0,t)}\right)$ where $\mathcal{K}_H^*$ is given by (16). Thus we have a formal series

$$W(t) = \sum_{n=1}^{\infty} w_n(t) e_n. \tag{50}$$

Stochastic integral

Let $(e_n, n \in \mathbf{N})$ be a complete orthonormal basis in U.
Let $G : [0, T] \to \mathcal{L}(U, V)$ be an operator-valued function such that $G(\cdot)e_n \in \mathcal{H}$ for $n \in \mathbf{N}$, and B^H be a standard cylindrical fractional Brownian motion in U.
Define

$$\int_0^T G \, dB^H := \sum_{n=1}^{\infty} \int_0^T Ge_n \, d\beta_n^H$$

provided the infinite series converges in $L^2(\Omega, V)$.
Note that by condition 2 in the definition above the scalar processes $\beta_n^H(t) := \left\langle B^H(t), e_n \right\rangle, t \in \mathbf{R}, n \in \mathbf{N}$ are independent.

Linear equations

Consider the linear equation

$$\begin{aligned} dZ^x(t) &= AZ^x(t)\,dt + \Phi\,dB^H(t), \\ Z(0) &= x, \end{aligned} \tag{51}$$

where $(B^H(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (0,1)$ in U and U is a separable Hilbert space, $A : \mathrm{Dom}(A) \to V$, $\mathrm{Dom}(A) \subset V$, A is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on V, $\Phi \in \mathcal{L}(U, V)$ and $x \in V$ is generally random. Let $Q = \Phi\Phi^* \in \mathcal{L}(V)$.

Linear equations

A solution $(Z^x(t), t \geq 0)$ to (51) is considered in the mild form

$$Z^x(t) = S(t)x + Z(t), \quad t \geq 0, \tag{52}$$

where $(Z(t), t \geq 0)$ is the convolution integral

$$Z(t) = \int_0^t S(t-u)\Phi \, dB^H(u). \tag{53}$$

If $(S(t), t \geq 0)$ is analytic, then there is a $\hat{\beta} \in \mathbf{R}$ such that the operator $\hat{\beta}I - A$ is uniformly positive on V.
For each $\delta \geq 0$, let us define $(V_\delta, |\cdot|_\delta)$ a Banach space, where $V_\delta = \mathrm{Dom}\left((\hat{\beta}I - A)^\delta\right)$ with the graph norm topology such that

$$|x|_\delta = \left|(\hat{\beta}I - A)^\delta x\right|_V .$$

The space V_δ does not depend on $\hat{\beta}$ because the norms are equivalent for different values of $\hat{\beta}$ satisfying the above condition.

Assumptions

Let $(S(t), t \geq 0)$ be an analytic semigroup such that

$$|S(t)\Phi|_\gamma \leq ct^{-\rho} \tag{A1}$$

for $t \in [0, T]$, $c \geq 0$ and $\rho \in [0, H)$.

Regularity

Theorem

If (A1) is satisfied, then $(Z(t), t \in [0, T])$ is a well-defined V_δ-valued process in $\mathcal{C}^\beta([0, T], V_\delta)$, a.s.-P for $\beta + \delta + \gamma < H, \beta \geq 0, \delta \geq 0$.

- Analyticity not necessary for $H > 1/2$.

Conjecture: Consider the general case $B_t = \sum \beta_n(t)$ where β_n are continuous centered Gaussian processes defined by (the same) kernel K satisfying (K1). Then the stochastic convolution integral exists and as a process has a version with sample paths in $L^2(0, T; V)$ a.s. provided (A1) is satisfied with $\rho = 0$. If moreover we have for some $H > 1/2$

$$\frac{\partial K}{\partial t}(t, s) \leq (s/t)^{1/2-H}(t-s)^{H-3/2}$$

the same holds true under weaker condition $\rho < H$.