Risk Aggregation with Dependence Uncertainty

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Motivation on VaR aggregation with dependence uncertainty

Full information on **marginal distributions**:

$X_j \sim F_j$

$+$

Full Information on **dependence**:

(known copula)

$\Rightarrow$

$\mathrm{VaR}_q\left(X_1 + X_2 + ... + X_n\right)$ can be computed!

Motivation on VaR aggregation with dependence uncertainty

Full information on **marginal distributions**:

$$X_j \sim F_j$$

$$+$$

Partial or **no** Information on **dependence**:
(incomplete information on copula)

$$\Rightarrow$$

$\mathrm{VaR}_q(X_1 + X_2 + ... + X_n)$ **cannot** be computed!

Only a range of possible values for $\mathrm{VaR}_q(X_1 + X_2 + ... + X_n)$.

Model Risk

1. Goal: Assess the risk of a portfolio sum $S = \sum_{i=1}^{d} X_i$.
2. Choose a risk measure $\rho(\cdot)$: variance, Value-at-Risk...
3. "Fit" a multivariate distribution for $(X_1, X_2, ..., X_d)$ and compute $\rho(S)$
4. How about model risk? How wrong can we be?

Model Risk

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4. How about model risk? How wrong can we be?

Assume $\rho(S) = var(S)$,

$$\rho_{\mathcal{F}}^{+} := \sup\left\{ var\left(\sum_{i=1}^{d} X_i\right)\right\}, \quad \rho_{\mathcal{F}}^{-} := \inf\left\{ var\left(\sum_{i=1}^{d} X_i\right)\right\}$$

where the bounds are taken over all other (joint distributions of) random vectors $(X_1, X_2, ..., X_d)$ that "agree" with the available information $\mathcal{F}$

Aggregation with dependence uncertainty: Example - Credit Risk

- Marginals known:
- Dependence fully unknown

Consider a portfolio of 10,000 loans all having a default probability $p = 0.049$. The default correlation is $\rho = 0.0157$ (for KMV).

	KMV VaR_q	Max VaR_q	Min VaR_q
q = 0.95	10.1%	98%	0%
q = 0.995	15.1%	100%	4.4%

Portfolio models are subject to significant model uncertainty (defaults are rare and correlated events).
Using dependence information is crucial to try to get more "reasonable" bounds.

Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of d dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?

Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of d dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?
- Implications:
  - Current VaR based regulation is subject to high model risk, even if one knows the multivariate distribution "almost completely".

Acknowledgement of Collaboration

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- Bernard, C., Rüschendorf, L., Vanduffel, S. (2015). Value-at-Risk bounds with variance constraints. *Journal of Risk and Insurance*
- Bernard, C., Vanduffel, S. (2015). A new approach to assessing model risk in high dimensions. *Journal of Banking and Finance*
- Bernard, C. , Rüschendorf, L., Vanduffel, S., Yao, J. (2015). How robust is the Value-at-Risk of credit risk portfolios? *European Journal of Finance*
- Bernard, C., X. Jiang, R. Wang, (2013) Risk Aggregation with Dependence Uncertainty, *Insurance: Mathematics and Economics*

Risk Aggregation and full dependence uncertainty

- Marginals known:
- Dependence fully unknown
- In two dimensions $d = 2$, assessing model risk on variance is linked to the Fréchet-Hoeffding bounds

$$var(F_1^{-1}(U)+F_2^{-1}(1-U)) \leqslant var(X_1+X_2) \leqslant var(F_1^{-1}(U)+F_2^{-1}(U))$$

- A challenging problem in $d \geqslant 3$ dimensions
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
  - Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR
- **Issues**
  - bounds are generally very wide
  - ignore all information on dependence.

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- **Issues**
  - bounds are generally very wide
  - ignore all information on dependence.
- **Our answer:**
  - incorporating in a natural way dependence information.

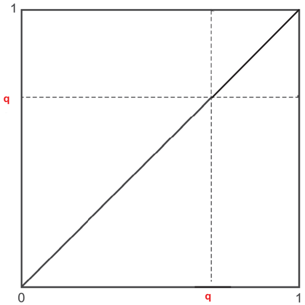
VaR Bounds with full dependence uncertainty

(Unconstrained VaR bounds)

"Riskiest" Dependence: maximum VaR_q in 2 dims

If X_1 and X_2 are U(0,1) comonotonic, then

$$VaR_q(S^c) = VaR_q(X_1) + VaR_q(X_2) = 2q.$$



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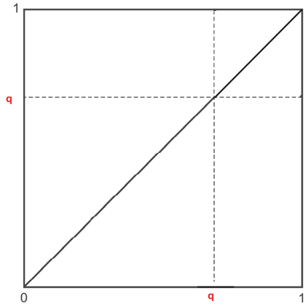
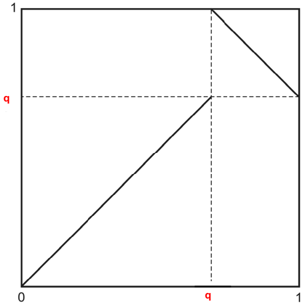


Note that $TVaR_q)(S^c) = \int_q^1 2pdp/(1-q) = 1+q.$

"Riskiest" Dependence: maximum VaR_q in 2 dims

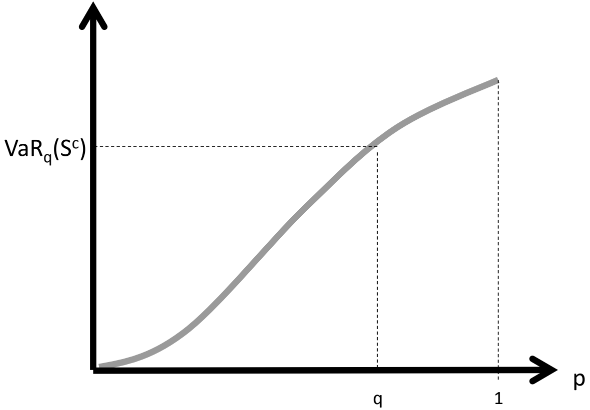
If X_1 and X_2 are U(0,1) and antimonotonic in the tail, then $VaR_q(S^*) = 1 + q$.



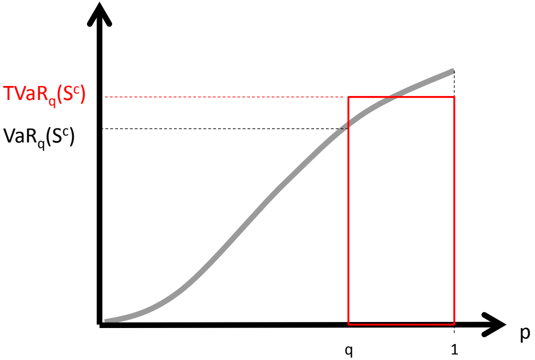
$$VaR_q(S^*) = 1 + q > VaR_q(S^c) = 2q$$

$\Rightarrow$ to maximize VaR_q, the idea is to change the comonotonic dependence such that the sum is constant in the tail

VaR at level q of the comonotonic sum w.r.t. q

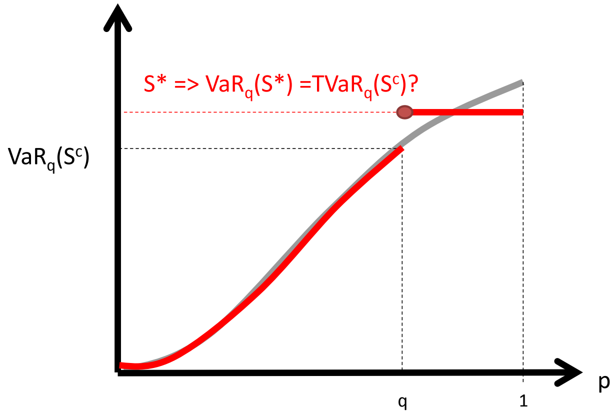


VaR at level q of the comonotonic sum w.r.t. q



where TVaR (Expected shortfall): $\text{TVaR}_q(X) = \frac{1}{1-q}\int_q^1 \text{VaR}_u(X)\mathrm{d}u$

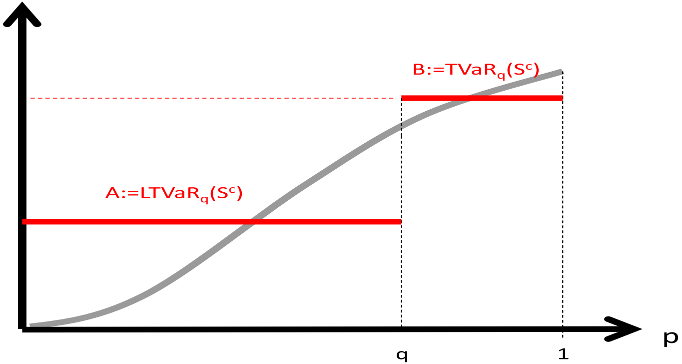
Riskiest Dependence Structure VaR at level q



Analytic expressions (not sharp)

Analytical Unconstrained Bounds with $X_j \sim F_j$

$$A = LTVaR_q(S^c) \leqslant \mathrm{VaR}_q\left[X_1 + X_2 + \ldots + X_n\right] \leqslant B = TVaR_q(S^c)$$



VaR Bounds with full dependence uncertainty

Approximate sharp bounds:

- Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
- Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR

Illustration for the maximum VaR (1/3)

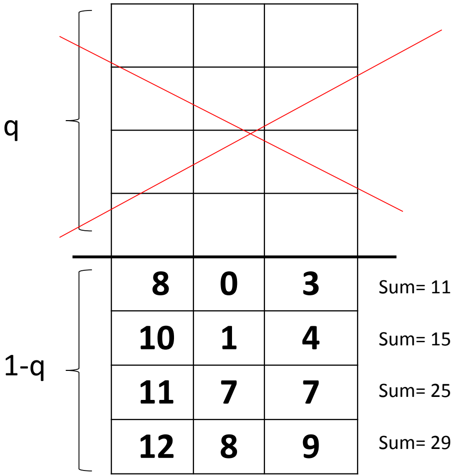


Illustration for the maximum VaR (2/3)

q

1-q

8	0	3	Sum= 11
10	1	4	Sum= 15
11	7	7	Sum= 25
12	8	9	Sum= 29

Rearrange **within** columns..to make the sums as constant as possible...

B=(11+15+25+29)/4=20

Illustration for the maximum VaR (3/3)

q				
1-q	**8**	**8**	**4**	Sum= 20
	10	**7**	**3**	Sum= 20
	12	**1**	**7**	Sum= 20
	11	**0**	**9**	Sum= 20

=B!

VaR Bounds with partial dependence uncertainty

VaR Bounds with Dependence Information...

Adding dependence information

Finding minimum and maximum possible values for VaR of the credit portfolio loss, $S = \sum_{i=1}^{n} X_i$, given that

- known marginal distributions of the risks X_i.
- some **dependence information.**

Example 1: variance constraint - with Rüschendorf and Vanduffel

$$M := \sup \mathrm{VaR}_q \left[X_1 + X_2 + ... + X_n\right],$$
$$\text{subject to } \; X_j \sim F_j, \mathrm{var}(X_1 + X_2 + ... + X_n) \leqslant s^2$$

Example 2: Moments constraint - with Denuit, Rüschendorf, Vanduffel, Yao

$$M := \sup \mathrm{VaR}_q \left[X_1 + X_2 + ... + X_n\right],$$
$$\text{subject to } \; X_j \sim F_j, \mathrm{E}(X_1 + X_2 + ... + X_n)^k \leqslant c_k$$

Adding dependence information

Example 3: VaR bounds when the joint distribution of $(X_1, X_2, ..., X_n)$ is known on a subset of the sample space: with Vanduffel.

Example 4: with Rüschendorf, Vanduffel and Wang

$$\begin{aligned} &M := \sup \mathrm{VaR}_q \left[X_1 + X_2 + ... + X_n\right], \\ &\text{subject to } \ (X_j, Z) \sim H_j, \end{aligned}$$

where Z is a factor.

Examples 1 and 2

Example 1: variance constraint

$$M := \sup \mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right],$$
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Example 2: Moments constraint

$$M := \sup \mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right],$$
$$\text{subject to } \; X_j \sim F_j, \mathrm{E}(X_1 + X_2 + ... + X_n)^k \leqslant c_k$$

for all k in 2,...,K

VaR bounds with moment constraints

- Without moment constraints, VaR bounds are attained if there exists a dependence among risks X_i such that

$$S = \begin{cases} A & \text{probability } q \\ B & \text{probability } 1-q \end{cases} \quad \text{a.s.}$$

  - If the "distance" between A and B is too wide then improved bounds are obtained with

$$S^* = \begin{cases} a & \text{with probability } q \\ b & \text{with probability } 1-q \end{cases}$$

such that

$$\begin{cases} a^k q + b^k (1-q) \leqslant c_k \\ aq + b(1-q) = E[S] \end{cases}$$

in which a and b are "as distant as possible while satisfying all constraints" (for all k)

Corporate portfolio

- a corporate portfolio of a major European Bank.
- 4495 loans mainly to medium sized and large corporate clients
- total exposure (EAD) is 18642.7 (million Euros), and the top 10% of the portfolio (in terms of EAD) accounts for 70.1% of it.
- portfolio exhibits some heterogeneity.

Summary statistics of a corporate portfolio

	Minimum	Maximum	Average
Default probability	0.0001	0.15	0.0119
EAD	0	750.2	116.7
LGD	0	0.90	0.41

Comparison of Industry Models

VaRs of a corporate portfolio under different industry models					
	$q =$	Comon.	KMV	Credit Risk$^+$	Beta
	95%	393.5	281.3	281.8	282.5
	95%	393.5	340.6	346.2	347.4
$\rho = 0.10$	99%	2374.1	539.4	513.4	520.2
	99.5%	5088.5	631.5	582.9	593.5

VaR bounds

With $\rho = 0.1$,

VaR assessment of a corporate portfolio					
$q =$	KMV	Comon.	Unconstrained	$K = 2$	$K = 3$
95%	340.6	393.3	(34.0 ; 2083.3)	(97.3 ; 614.8)	(100.9 ; 562.8)
99%	539.4	2374.1	(56.5 ; 6973.1)	(111.8 ; 1245)	(115.0 ; 941.2)
99.5%	631.5	5088.5	(89.4 ; 10120)	(114.9 ; 1709)	(117.6 ; 1177.8)

- Obs 1: Comparison with analytical bounds
- Obs 2: Significant bounds reduction with moments information
- Obs 3: Significant model risk

Example 3

Example 3: VaR bounds when the joint distribution of $(X_1, X_2, ..., X_n)$ is known on a subset of the sample space.

Bounds on variance

Analytical Bounds on Standard Deviation

Consider d risks X_i with standard deviation σ_i

$$0 \leqslant std(X_1 + X_2 + ... + X_d) \leqslant \sigma_1 + \sigma_2 + ... + \sigma_d$$

Bounds on variance

Analytical Bounds on Standard Deviation

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$$0 \leqslant std(X_1 + X_2 + ... + X_d) \leqslant \sigma_1 + \sigma_2 + ... + \sigma_d$$

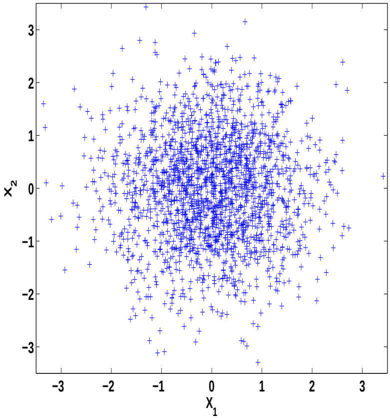
Example with 20 standard normal N(0,1)

$$0 \leqslant std(X_1 + X_2 + ... + X_{20}) \leqslant 20$$

and in this case, both bounds are sharp but too wide for practical use!

Our idea: Incorporate information on dependence.

Illustration with 2 risks with marginals N(0,1)



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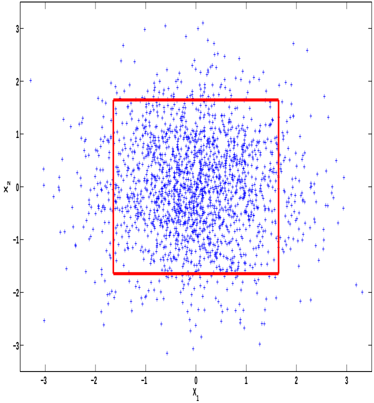
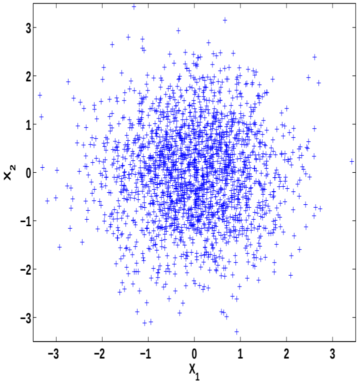


Assumption: Independence on $\mathcal{F} = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$

Illustration with marginals N(0,1)



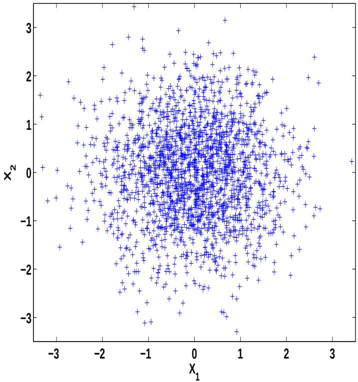
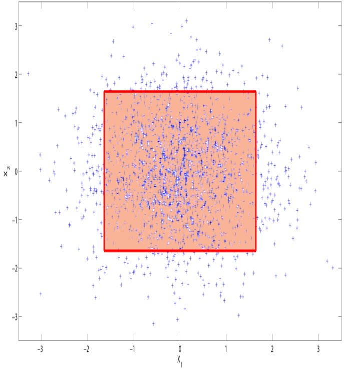
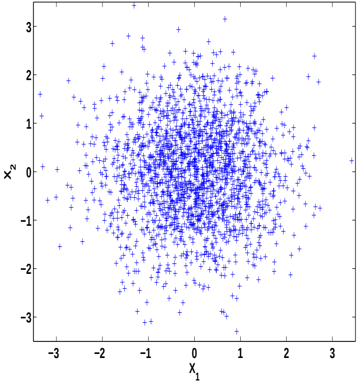


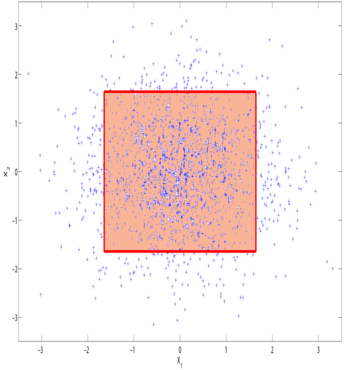
Illustration with marginals N(0,1)





$$\mathcal{F}_1 = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$$

Illustration with marginals N(0,1)



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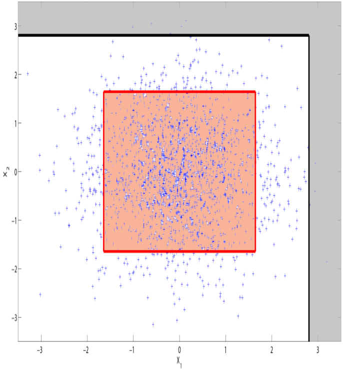
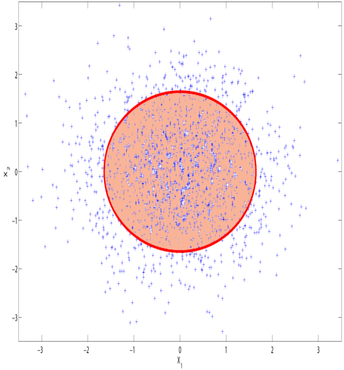
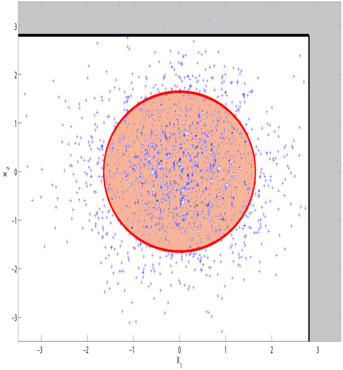


$$\mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1$$

Illustration with marginals N(0,1)





$\mathcal{F}_1$ =contour of MVN at β

$$\mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1$$

Our assumptions on the cdf of $(X_1, X_2, ..., X_d)$

$\mathcal{F} \subset \mathbb{R}^d$ ("trusted" or "fixed" area)
$\mathcal{U} = \mathbb{R}^d \backslash \mathcal{F}$ ("untrusted").

We assume that we know:

(i) the marginal distribution F_i of X_i on $\mathbb{R}$ for $i = 1, 2, ..., d$,
(ii) the distribution of $(X_1, X_2, ..., X_d) \mid \{(X_1, X_2, ..., X_d) \in \mathcal{F}\}$.
(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

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(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

- When only marginals are known: $\mathcal{U} = \mathbb{R}^d$ and $\mathcal{F} = \emptyset$.
- **Our Goal: Find bounds on** $var(S) := var(X_1 + ... + X_d)$ **when** $(X_1, ..., X_d)$ **satisfy (i), (ii) and (iii).**

Example $d = 20$ risks N(0,1)

- $(X_1, ..., X_{20})$ correlated N(0,1) on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \qquad p_f = P\left((X_1, ..., X_{20}) \in \mathcal{F}\right)$$

(for some $\beta \leqslant 50\%$) where q_γ: γ-quantile of N(0,1)

- $\beta = 0\%$: no uncertainty (20 correlated N(0,1))
- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$			$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47			(0 , 20)

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- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47	(4.4 , 5.65)	(3.89 , 10.6)	(0 , 20)

Model risk on the volatility of a portfolio is reduced a lot by incorporating information on dependence!

Numerical Results for VaR, 20 risks $N(0,1)$

When marginal distributions are given,

- What is the maximum Value-at-Risk?
- What is the minimum Value-at-Risk?
- A portfolio of 20 risks normally distributed N(0,1). Bounds on VaR_q (by the rearrangement algorithm applied on each tail)

q=95%	(-2.17 , 41.3)
q=99.95%	(-0.035 , 71.1)

- More examples in Embrechts, Puccetti, and Rüschendorf (2013): "Model uncertainty and VaR aggregation," *Journal of Banking and Finance*
- Very wide bounds
- All dependence information ignored

Idea: add information on dependence from a fitted model where data is available...

Numerical Results, 20 correlated $N(0,1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

$\mathcal{F}$	$\mathcal{U} = \emptyset$ $\beta = 0\%$			$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
q=95%	12.5			(-2.17 , 41.3)
q=99.95%	25.1			(-0.035 , 71.1)

- $\mathcal{U} = \emptyset$: 20 correlated standard normal variables ($\rho = 0.1$).

$$\text{VaR}_{95\%} = 12.5 \qquad \text{VaR}_{99.95\%} = 25.1$$

Numerical Results, 20 correlated $N(0,1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

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q=95%	12.5	(12.2 , 13.3)	(10.7 , 27.7)	(-2.17 , 41.3)
q=99.95%	25.1	(24.2 , 71.1)	(21.5 , 71.1)	(-0.035 , 71.1)

- $\mathcal{U} = \emptyset$: 20 correlated standard normal variables ($\rho = 0.1$).

$$\text{VaR}_{95\%} = 12.5 \quad \text{VaR}_{99.95\%} = 25.1$$

- **The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.**
- **For VaR at high probability levels ($q = 99.95\%$), despite all the added information on dependence, the bounds are still wide!**

Conclusions

We have shown that

- Maximum Value-at-Risk is not caused by the comonotonic scenario.
- Maximum Value-at-Risk is achieved when the variance is *minimum* in the tail. The RA is then used in the tails only.
- Bounds on Value-at-Risk at high confidence level stay wide even if the multivariate dependence is known in 98% of the space!

- Assess model risk with partial information and given marginals
- Design algorithms for bounds on variance, TVaR and VaR and many more risk measures.
- Challenges:
  - How to choose the trusted area $\mathcal{F}$ optimally?
  - Re-discretizing using the fitted marginal $\hat{f}_i$ to increase N
  - Incorporate uncertainty on marginals

Regulation challenge

The Basel Committee (2013) insists that **a desired objective of a Solvency framework concerns comparability**:

"*Two banks with portfolios having identical risk profiles apply the framework's rules and arrive at the same amount of risk-weighted assets, and two banks with different risk profiles should produce risk numbers that are different proportionally to the differences in risk*"

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References

- Bernard, C., Vanduffel S. (2015): "A new approach to assessing model risk in high dimensions", *Journal of Banking and Finance.*
- Bernard, C., M. Denuit, and S. Vanduffel (2014): "Measuring Portfolio Risk under Partial Dependence Information," *Working Paper*.
- Bernard, C., X. Jiang, and R. Wang (2014): "Risk Aggregation with Dependence Uncertainty," *Insurance: Mathematics and Economics*.
- Bernard, C., L. Rüschendorf, and S. Vanduffel (2014): "VaR Bounds with a Variance Constraint," *Journal of Risk and Insurance*.
- Embrechts, P., G. Puccetti, and L. Rüschendorf (2013): "Model uncertainty and VaR aggregation," *Journal of Banking & Finance*.
- Puccetti, G., and L. Rüschendorf (2012): "Computation of sharp bounds on the distribution of a function of dependent risks," *Journal of Computational and Applied Mathematics*, 236(7), 1833–1840.
- Wang, B., and R. Wang (2011): "The complete mixability and convex minimization problems with monotone marginal densities," *Journal of Multivariate Analysis*, 102(10), 1344–1360.
- Wang, B., and R. Wang (2015): "Joint Mixability," *Mathematics Operational Research*.