Equilibrium in risk-sharing games

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joint with
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Motivation

- Financial agents share their risky positions by designing new (or trading given) financial securities in a mutually beneficial way.
- These transactions are normally **not cooperative**. They involve only a small number of agents, each of which can influence the equilibrium.
- Agents' **strategic behaviour** in risk sharing should be introduced.

We ask:

✓ *How should an agent respond to the actions of the others?* (Best response problem)

✓ *How and at which point the market equilibrate?* (Nash equilibrium)

✓ *Do certain agents benefit from the game?* (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
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Agents and preferences

Static probability model

- $\mathbb{L}^0 \equiv \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$: *discounted* future payoffs.
- $I = \{0, \ldots, n\}$: index set of $n+1$ economic agents.

Preferences

- Agents' risk preferences modelled via *monetary* utility functionals:

$$\mathbb{L}^0 \ni X \mapsto \mathbb{U}_i(X) := -\delta_i \log\left(\mathbb{E}\left[\exp\left(-\frac{X}{\delta_i}\right)\right]\right) \in [-\infty, \infty).$$

- Define the aggregate risk tolerance $\delta := \sum_{i \in I} \delta_i$, as well as

$$\lambda_i := \frac{\delta_i}{\delta}, \quad \delta_{-i} := \delta - \delta_i, \quad \forall i \in I.$$

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Endowments and Securities

Endowments

- $E_i \in \mathbb{L}^0$: random endowment (risky position) of agent $i \in I$.
- Aggregate endowment:

$$E := \sum_{i \in I} E_i.$$

- **Standing assumption** enforced throughout: $(E_i)_{i \in I} \in \mathcal{E}$; in effect,

$$\mathbb{U}_i(E_i) > -\infty, \quad \forall i \in I.$$

Sharing-Securities-Valuation measure

A risk sharing transaction consists of a valuation measure $\mathbb{Q} \in \mathcal{P}$ and a collection of security payoffs $(C_i)_{i \in I}$ belonging in the following set:

$$\mathcal{C}_{\mathbb{Q}} := \left\{ (C_i)_{i \in I} \in (\mathbb{L}^0)^I \;\middle|\; \sum_{i \in I} C_i = 0, \; C_i \in \mathbb{L}^1(\mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}}[C_i] = 0, \; \forall i \in I \right\}.$$

$\rightarrow$ After sharing, position of agent $i \in I$ is $E_i + C_i$.

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Complete market equilibrium

Arrow-Debreu equilibrium

Valuation probability $\mathbb{Q}^* \in \mathcal{P}$ and securities $(C_i^*)_{i \in I} \in (\mathbb{L}^0)^I$ such that:

- $(C_i^*)_{i \in I} \in \mathcal{C}_{\mathbb{Q}^*}$.
- For all $C \in \mathbb{L}^1(\mathbb{Q}^*)$ with $\mathbb{E}_{\mathbb{Q}^*}[C] \leq 0$, $\mathbb{U}_i(E_i + C) \leq \mathbb{U}_i(E_i + C_i^*)$, $\forall i \in I$.

Theorem (Borch '62)

A unique Arrow-Debreu equilibrium exists; in fact, $\mathrm{d}\mathbb{Q}^*/\mathrm{d}\mathbb{P} \propto \exp(-E/\delta)$ and

$$C_i^* := \lambda_i E - E_i - \mathbb{E}_{\mathbb{Q}^*}[\lambda_i E - E_i], \quad \forall i \in I.$$

Aggregate monetary utility in Arrow-Debreu equilibrium

$(C_i^*)_{i \in I}$ is a maximiser of $\sum_{i \in I} \mathbb{U}_i(E_i + C_i)$; furthermore,

$$\sum_{i \in I} \mathbb{U}_i(E_i + C_i^*) = -\delta \log \mathbb{E}[\exp(-E/\delta)] \geq \sum_{i \in I} \mathbb{U}_i(E_i).$$

$\rightarrow$ "$\geq$" above is "$=$" $\iff$ $C_i^* = 0$, $\forall i \in I$.

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Outline

Reported endowments

Agents may have motive to report different endowments than their actual ones.

Stage 1: Agents agree on the sharing rules of the *reported* endowments.

What if instead of $(E_i)_{i\in I} \in \mathcal{E}$, agents choose to report $(F_i)_{i\in I} \in \mathcal{E}$?

- With $F := \sum_{i\in I} F_i$, the valuation measure $\mathbb{Q}^F$ is such that

$$\mathrm{d}\mathbb{Q}^F/\mathrm{d}\mathbb{P} \propto \exp\left(-F/\delta\right).$$

- Leads to risk sharing with securities

$$\begin{aligned} C_i &= \lambda_i F - F_i - \mathbb{E}_{\mathbb{Q}^F}\left[\lambda_i F - F_i\right] \\ &= \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i}}\left[\lambda_i F_{-i} - \lambda_{-i} F_i\right], \quad \forall i \in I. \qquad (\star) \end{aligned}$$

Revealed endowments via valuation measure and securities

Given $\mathbb{Q}$ and $(C_i)_{i\in I} \in \mathcal{C}_{\mathbb{Q}}$, $\exists (F_i)_{i\in I}$ (unique up to cash translation) such that

$$\mathbb{Q} = \mathbb{Q}^F \quad \text{and} \quad (C_i)_{i\in I} \quad \text{are given by } (\star).$$

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Best endowment response: the problem

Consider the position of agent $i \in I$. Given

- the agreed mechanism that produces the optimal sharing securities; and
- the endowment F_{-i} reported by the rest n agents in $I \setminus \{i\}$,

a natural question is:

Which random quantity should agent $i \in I$ report as actual endowment?

Response function

Let F_{-i} given. The **response function** of agent $i \in I$ is

$$\mathbb{V}_i(F_i; F_{-i}) := \mathbb{U}_i\left(E_i + \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i}}\left[\lambda_i F_{-i} - \lambda_{-i} F_i\right]\right).$$

- $\mathbb{V}_i(F_i + c; F_{-i}) = \mathbb{V}_i(F_i; F_{-i})$ holds for all $c \in \mathbb{R}$.
- $\mathbb{V}_i(\cdot; F_{-i})$ is *not* concave in general.

Best response

For given F_{-i}, we seek F_i^r such that

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Best endowment response: results

Proposition (Necessary and sufficient conditions for optimality)

Let $i \in I$, F_{-i} and F_i^{r} given. The following are equivalent:

1. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.
2. $C_i^{\mathrm{r}} := \lambda_i F_{-i} - \lambda_{-i} F_i^{\mathrm{r}} - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i^{\mathrm{r}}}}\left[\lambda_i F_{-i} - \lambda_{-i} F_i^{\mathrm{r}}\right]$ is such that

$$\frac{\delta}{\delta_{-i}} \frac{C_i^{\mathrm{r}}}{\delta_i} + \log\left(1 + \frac{C_i^{\mathrm{r}}}{\delta_{-i}}\right) = \zeta_i - \frac{E_i}{\delta_i} + \frac{F_{-i}}{\delta_{-i}},$$

(note the *a-priori* necessary bound $C_i^{\mathrm{r}} > -\delta_{-i}$) and $\zeta_i \in \mathbb{R}$ is such that

$$\zeta_i = \frac{\mathbb{U}_i(E_i + C_i^{\mathrm{r}})}{\delta_i} - \frac{\mathbb{U}_i\left(F_{-i} - C_i^{\mathrm{r}}\right)}{\delta_{-i}}.$$

(1) $\Rightarrow$ (2): 1st-order conditions. $\mathbb{V}_i(\cdot; F_{-i})$ is not concave: (2) $\Rightarrow$ (1) is tricky.

Theorem

There exists unique (up to constants) F_i^{r} s.t. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.

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(note the *a-priori* necessary bound $C_i^{\mathrm{r}} > -\delta_{-i}$) and $\zeta_i \in \mathbb{R}$ is such that

$$\zeta_i = \frac{\mathbb{U}_i(E_i + C_i^{\mathrm{r}})}{\delta_i} - \frac{\mathbb{U}_i\left(F_{-i} - C_i^{\mathrm{r}}\right)}{\delta_{-i}}.$$

(1) $\Rightarrow$ (2): 1st-order conditions. $\mathbb{V}_i(\cdot; F_{-i})$ is not concave: (2) $\Rightarrow$ (1) is tricky.

Theorem

There exists unique (up to constants) F_i^{r} s.t. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.

Best endowment response: results

Proposition (Necessary and sufficient conditions for optimality)

Let $i \in I$, F_{-i} and F_i^{r} given. The following are equivalent:

1. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.
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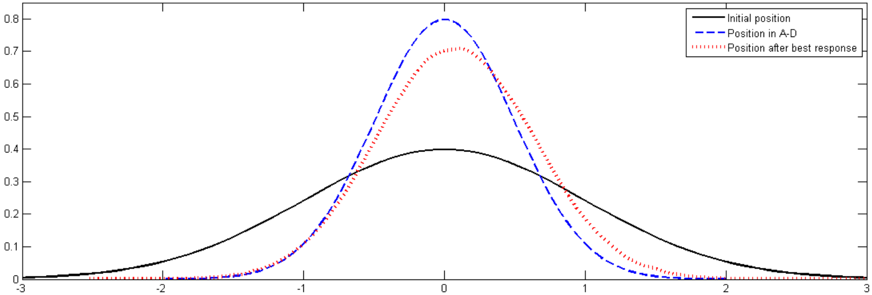
An illustrative example



Two-agent example, where endowments are correlated ($\rho = -0.2$) and normal distributed, $\delta_i = 1$ for $i = 0, 1$.

Outline

Nash Equilibrium

Stage 2

- All agents have same strategic behaviour.
- Given the agreed risk sharing rules (stage 1), agents negotiate the securities they are going to trade and the valuation measure they are going to apply.

Definition

The pair $(\mathbb{Q}^\diamond, (C_i^\diamond)_{i\in I}) \in \mathbb{P} \times (\mathbb{L}^0)^I$ will be called a **Nash risk sharing equilibrium** if

$$\mathbb{V}_i\left(F_i^\diamond; F_{-i}^\diamond\right) = \sup_{F_i} \mathbb{V}_i\left(F_i; F_{-i}^\diamond\right), \quad \forall i \in I,$$

where $(F_i^\diamond)_{i\in I}$ are the corresponding revealed endowments, given implicitly by

$$\frac{\mathrm{d}\mathbb{Q}^\diamond}{\mathrm{d}\mathbb{P}} \propto \exp\left(-F^\diamond/\delta\right)$$

and

$$C_i^\diamond = \lambda_i F^\diamond - F_i^\diamond - \mathbb{E}_{\mathbb{Q}^\diamond}\left[\lambda_i F^\diamond - F_i^\diamond\right].$$

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Necessary and sufficient conditions for Nash equilibrium

Theorem

The collection $(\mathbb{Q}^\diamond, (C_i^\diamond)_{i \in I}) \in \mathbb{P} \times (\mathbb{L}^0)^I$ is a Nash equilibrium if and only if the following three conditions hold:

1. $C_i^\diamond > -\delta_{-i}$ for all $i \in I$, and there exists $z^\diamond \equiv (z_i^\diamond)_{i \in I} \in \mathbb{R}^I$ with $\sum_{i \in I} z_i^\diamond = 0$ such that

$$C_i^\diamond + \delta_i \log \left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right) = z_i^\diamond + C_i^* + \lambda_i \sum_{j \in I} \delta_j \log \left(1 + \frac{C_j^\diamond}{\delta_{-j}}\right), \quad \forall i \in I.$$

2. $\frac{\mathrm{d}\mathbb{Q}^\diamond}{\mathrm{d}\mathbb{Q}^*} \propto -\sum_{j \in I} \lambda_j \log \left(1 + \frac{C_j^\diamond}{\delta_{-j}}\right)$.
3. $\mathbb{E}_{\mathbb{Q}^\diamond} [C_i^\diamond] = 0$ holds for all $i \in I$.

Existence (and uniqueness) of Nash equilibria?

In search of equilibrium

Parametrise candidate optimal securities in

$$\Delta^I := \left\{ (z_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} z_i = 0 \right\} \equiv \mathbb{R}^n \quad \text{(where } n = \#I - 1\text{)}.$$

- For all $z \in \Delta^I$, $\exists!$ $(C_i(z))_{i \in I}$ with $\sum_{i \in I} C_i(z) = 0$ satisfying equations (1).
- Aim: find $z \in \Delta^I$ such that $\mathbb{E}_{\mathbb{Q}(z)}[C_i(z)] = 0$ holds for all $i \in I$.

Theorem

1. In a Nash equilibrium, $\mathbb{E}_{\mathbb{Q}(z^\diamond)}[C_i(z^\diamond)] = 0$ holds $\forall i \in I$.
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Theorem

If $I = \{0, 1\}$, there exists a unique $z^\diamond \in \Delta^I \equiv \mathbb{R}$ with $\mathbb{E}_{\mathbb{Q}(z^\diamond)}[C_i(z^\diamond)] = 0$, $\forall i \in I$.

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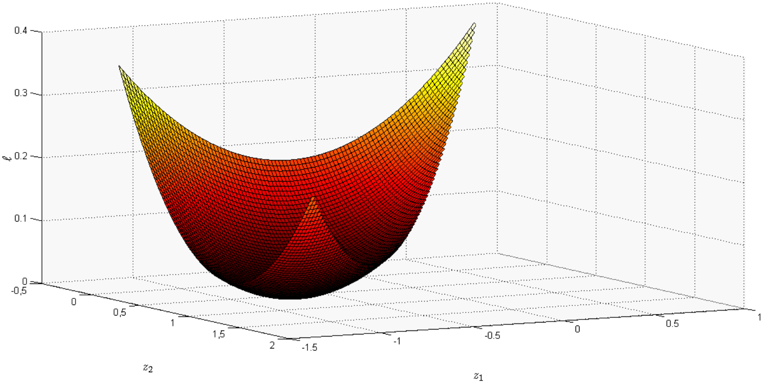
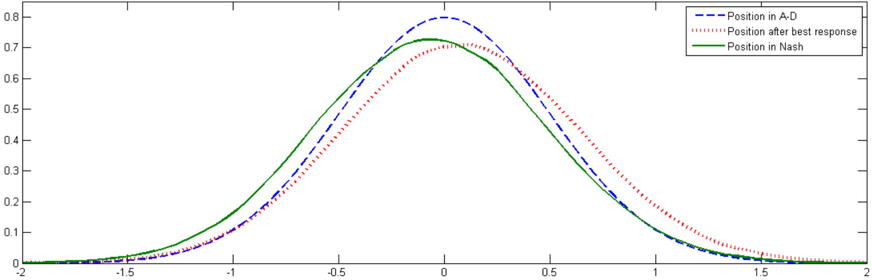


Three-agent example, where endowments are correlated and normal distributed, $\delta_i = 1$ for $i = 0, 1, 2$ and

$$\text{Distance}(z) = \sum_{i=0}^{2} -\delta_{-i} \log \left(1 + \frac{\mathbb{E}_{\mathbb{Q}(z)}\left[C_i(z)\right]}{\delta_{-i}} \right), \quad z \in \Delta^I.$$

A two-agent example



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Some consequences of Nash equilibrium

You trade, you share endowment different than your true one

$$F_i^\diamond = E_i - z_i^\diamond + \delta_i \log\left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right).$$

- For any fixed $i \in I$, $F_i^\diamond - E_i = \text{constant} \iff C_i^\diamond = 0 \iff C_i^* = 0.$

Endogenous bounds on securities

It holds that $C_i^\diamond > -\delta_{-i}$ for all $i \in I$. Hence,

$$-\delta_{-i} < C_i^\diamond < (n-1)\delta + \delta_i, \quad \forall i \in I. \quad \text{[Contrast with A-D equilibrium.]}$$

Aggregate loss of efficiency (in monetary terms)

$$\sum_{i \in I} \mathbb{U}_i(E_i + C_i^*) - \sum_{i \in I} \mathbb{U}_i(E_i + C_i^\diamond) = -\delta \log \mathbb{E}_{\mathbb{Q}^\diamond}\left[\prod_{i \in I}\left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right)^{\delta_i/\delta}\right] \geq 0.$$

No loss of efficiency $\iff C_i^* = 0, \ \forall i \in I \iff C_i^\diamond = 0, \ \forall i \in I.$

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What about the M-V preferences?

Let the agents' preferences be mean-variance ones:

$$\mathbb{L}^0 \ni X \mapsto \mathbb{U}_i(X) := \mathbb{E}[X] - \frac{1}{\delta_i}\mathbb{V}\text{ar}[X] \in [-\infty, \infty).$$

$\hookrightarrow$ Also in this case, the optimal sharing rules are of the form:

$$C_i^* := \lambda_i E - E_i - \mathbf{p}_i^*. \quad \text{[The CAPM.]}$$

Proposition

Under M-V preferences, the unique (up to constants) Nash risk sharing securities are given by

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for some constants α_i and α_{-i} and $\mathbf{p}_i^\diamond$ is a price vector.

- ✓ Just as the exponential case, $C_i^\diamond = C_i^*$ if and only if they are constants.
- ✓ Under the first order approximation $\log(x) \approx x - 1$, Nash risk sharing securities with M-V and exponential preferences are of the equal (up to constants).

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Outline

A sequence of markets

Set-up and notation

- Two agents: $I = \{0, 1\}$.
- A sequence of markets, indexed by $m \in \mathbb{N}$.
- $\delta_1^m \equiv \delta_1 \in (0, \infty)$ for all $m \in \mathbb{N}$, whereas $\lim_{m\to\infty} \delta_0^m = \infty$.
- E_0 and E_1 fixed.

Arrow-Debreu limit

- Limiting valuation measure $\mathbb{Q}^{\infty,*} = \mathbb{P}$.
- Limiting securities: $C_0^{\infty,*}$ and $C_1^{\infty,*} = -C_0^{\infty,*}$, with

$$C_0^{\infty,*} = E_1 - \mathbb{E}[E_1].$$

- Limiting utility gain (in monetary terms): with

$$u_i^{\infty,*} := \lim_{m\to\infty} \left(\mathbb{U}_i^m \left(E_i + C_i^{m,*} \right) - \mathbb{U}_i^m (E_i) \right), \quad \forall i \in \{0, 1\},$$

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Game limit

Limiting securities and valuation

- Limiting Nash-equilibrium security $C_0^{\infty,\diamond}$ for agent 0 satisfies

$$C_0^{\infty,\diamond} + \delta_1 \log\left(1 + \frac{C_0^{\infty,\diamond}}{\delta_1}\right) = z^{\infty,\diamond} + E_1,$$

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Limiting utility gain/loss (in monetary terms)

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Outline

Risk sharing in incomplete markets

The competitive prices

- The agents do not design new but **trade given security payoffs** in order to share their risky endowments.
- In **no** strategic behaviour case, each agent i submits his demand function on a given vector of securities $\mathbf{C} \in (\mathbb{L}^0)^k$

$$Z_i(\mathbf{p}) = \underset{\mathbf{a}\in\mathbb{R}^k}{\operatorname{argmax}} \left\{ \mathbb{U}_i(E_i + \mathbf{a}\cdot\mathbf{C} - \mathbf{a}\cdot\mathbf{p}) \right\}.$$

- The *(partially)* optimal equilibrium on $\mathbf{C}$ is a pair of prices and allocations $(p^*, A^*) \in \mathbb{R}^k \times \mathbb{R}^{(n+1)\times k}$ for which

$$Z_i(\mathbf{p}^*) = \mathbf{a}_i^*, \quad \forall i \in I,$$

where $\mathbf{a}_i^*$ denotes the i-th row of A^*.

✓ Under M-V this is the CAPM:

$$\mathbf{p}^* = \mathbb{E}[\mathbf{C}] - \frac{2}{\delta}\mathbb{C}\text{ov}(\mathbf{C}, E).$$

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Risk sharing in incomplete markets

The competitive prices

- The agents do not design new but **trade given security payoffs** in order to share their risky endowments.
- In **no** strategic behaviour case, each agent i submits his demand function on a given vector of securities $\mathbf{C} \in (\mathbb{L}^0)^k$

$$Z_i(\mathbf{p}) = \underset{\mathbf{a} \in \mathbb{R}^k}{\operatorname{argmax}} \{\mathbb{U}_i(E_i + \mathbf{a} \cdot \mathbf{C} - \mathbf{a} \cdot \mathbf{p})\}.$$

- The *(partially)* optimal equilibrium on $\mathbf{C}$ is a pair of prices and allocations $(p^*, A^*) \in \mathbb{R}^k \times \mathbb{R}^{(n+1)\times k}$ for which

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The game on demand functions

The preferable price vector

Given the aggregate demand submitted by the rest of the agents, agent i is going to respond a demand function that clears out the market at his preferable price:

$$\mathbf{p}_i^r := \operatorname*{argmax}_{\mathbf{p} \in \mathbb{R}^k} \{ \mathbb{U}_i(E_i - Z_{-i}(\mathbf{p}) \cdot \mathbf{C}) + Z_{-i}(\mathbf{p}) \cdot \mathbf{p} \}.$$

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Let $\mathcal{Z}_i$ be the set of all possible demand functions submitted by the agent i. Then, the *best demand response* of agent i is the demand function $Z_i^r \in \mathcal{Z}_i$ for which

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Define $\mathcal{Z} = \mathcal{Z}_0 \times ... \times \mathcal{Z}_n$.

Nash equilibrium in incomplete market

A pair $(\mathbf{p}^\diamond, (Z_i^\diamond)_{i \in I}) \in \mathbb{R}^k \times \mathcal{Z}$ is called Nash price-demand equilibrium of a vector of securities $\mathbf{C} \in (\mathbb{L}^0)^k$ if

$$\sum_{i \in I} Z_i^\diamond(\mathbf{p}^\diamond) = 0$$

and $\mathbf{p}^\diamond$ the preferable price for each agent, given the aggregate demand $Z^\diamond_{-i}$.

Under M-V preferences

- ✓ $\mathbf{p}^\diamond = \mathbb{E}[\mathbf{C}] - \frac{2}{\delta}\mathbb{C}\mathrm{ov}(\mathbf{C}, F^\diamond)$ *(oligopoly version of CAPM)*.
- ✓ $\mathbf{p}^\diamond = \mathbf{p}^*$ if and only if $\delta_i = \delta_j$, for all $i, j \in I$.
- ✓ For sufficiently low risk averse agents, $\mathbf{p}^\diamond$ is always better price than $\mathbf{p}^*$.

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Conclusive remarks & open questions

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- Such strategic behaviour gives an explanation of the **risk sharing inefficiency** and **security mispricing** that occur in markets with few agents.
- In Nash equilibrium, agents **never** choose to share their true risk exposure.
- In symmetric games, every agent suffers loss of utility as compared to the Arrow-Debreu equilibrium one.
- Strategic games **benefit** agents with **high risk tolerance**.

The future of risk sharing games...

- Existence (and uniqueness?) for more than two players.
- What about the presence of market makers in the transaction?
- Other risk-sharing rules?
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