Expected Shortfall revisited

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LUH-Kolloquium "Versicherungs- und Finanzmathematik"
The Future of Risk Measurement
Hannover, 11 December 2014
A few questions to start with

1. **Policyholder perspective:** Assume that for two companies with identical liabilities, policyholders get the same amounts in every state of the world.
   → Is it reasonable that the regulator deems one of these companies to be adequately capitalized but not the other? (≈ *surplus invariance*)

2. **Regulatory arbitrage:** Assume one company can choose from two jurisdictions with different regulators to set up the home office.
   → Is it reasonable that one regulator deems the company adequately capitalized while the other doesn’t? (≈ *numéraire invariance*)

3. **Harmonization of global regulation:** Assume regulators agreed globally on a single method for capital requirements in their respective jurisdictions.
   → Would Expected Shortfall (ES) in the relevant local currency be a good choice? What about Value-at-Risk (VaR)?
Objective of the presentation

- The objective of this presentation is to assess how ES performs in terms of surplus and numéraire invariance (formally introduced later)

- Our work complements current debate on ES vs. VaR which focuses exclusively on statistical properties (for an overview of this debate see [3])
Basic question in a capital adequacy framework

Starting point: At time 0 a financial institution selects a portfolio of assets and liabilities and at time $T$ assets are liquidated and liabilities repaid

→ Liability holders worry that the institution may default at time $T$, i.e. that capital ($= \text{"assets minus liabilities"}$) may become negative at time $T$, ...

→ ... but they are also unwilling to bear the costs of fully eliminating the risk of default and have to settle for some acceptable level of security

Key question for regulators: what is an acceptable level of security for policyholder liabilities, i.e. when should an insurer be deemed to be adequately capitalized?
Testing for capital adequacy: acceptance sets

*Capital position* of insurers, i.e. assets minus liabilities, at time $T$ are random variables $X : \Omega \to \mathbb{R}$ defined (for simplicity) on a finite state space $\Omega := \{\omega_1, \ldots, \omega_n\}$. $\mathcal{X}$ denotes the vector space of all possible capital positions

$$\rightarrow X(\omega) = \text{“value of assets less value of liabilities in state } \omega\text{”}$$

Regulators subject insurers to a *capital adequacy test* by checking whether their capital positions belong to an *acceptance set* $\mathcal{A} \subset \mathcal{X}$ satisfying two minimal requirements:

$$\rightarrow \text{Non-triviality: } \emptyset \neq \mathcal{A} \neq \mathcal{X}$$

$$\rightarrow \text{Monotonicity: } X \in \mathcal{A} \text{ and } Y \geq X \text{ imply } Y \in \mathcal{A}$$

Warning: We use interchangeably the terms acceptance set, capital adequacy test, acceptability criterion
The simplest acceptability criterion: scenario testing

The simplest acceptance criterion is testing whether an insurer can meet its obligations on a pre-specified set of states of the world \( A \subset \Omega \). The corresponding acceptance sets are called of \textit{SPAN-type} and given by

\[
\text{SPAN}(A) := \{ X \in \mathcal{X} ; X(\omega) \geq 0 \text{ for every } \omega \in A \}.
\]

\( \rightarrow \) SPAN stands for Standard Portfolio ANalysis.

\( \rightarrow \) SPAN\((A)\) is a closed, coherent acceptance set.

\( \rightarrow \) In the extreme case \( A = \Omega \), the set \( \text{SPAN}(A) \) coincides with the set of positive random variables, i.e. an insurer would be required to be able to pay claims in every state of the world!
The two most common acceptability criteria: $\text{VaR}_\alpha$ and $\text{ES}_\alpha$

The **Value-at-Risk acceptance set** at the level $0 < \alpha < 1$ is the closed, (generally) non-convex cone

$$A_\alpha := \{ X \in \mathcal{X} ; \ P(X < 0) \leq \alpha \} = \{ X \in \mathcal{X} ; \ \text{VaR}_\alpha(X) \leq 0 \},$$

where

$$\text{VaR}_\alpha(X) := \inf \{ m \in \mathbb{R} ; \ P(X + m < 0) \leq \alpha \}.$$ 

The **Expected Shortfall acceptance set** at the level $0 < \alpha < 1$ is closed and coherent and defined by

$$A^\alpha := \{ X \in \mathcal{X} ; \ \text{ES}_\alpha(X) \leq 0 \},$$

where

$$\text{ES}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) \, d\beta.$$
Surplus invariance introduced

Two insurers $X = A - L$ and $Y = A' - L$ with identical liabilities and possibly different assets. Policyholders get the same payments in all states of the world, i.e.

$$D_X = D_Y$$

where $D_X := \max\{-X, 0\}$ and $D_Y := \max\{-Y, 0\}$ are the options to default. Reasonable: $X$ and $Y$ should be either both acceptable or both unacceptable!

Definition ([5])
An acceptance set $\mathcal{A} \subset \mathcal{X}$ is said to be surplus invariant, if

$$X \in \mathcal{A}, \ Y \in \mathcal{X}, \ D_X = D_Y \implies Y \in \mathcal{A}.$$ 

We have $X = S_X - D_X$ where $S_X := \max\{X, 0\}$ is the surplus. Hence, $\mathcal{A}$ is surplus invariant if acceptability does not depend on the surplus.
Surplus invariance definition: too strong?

Stated in terms of capital positions $X = A - L$ and $Y = A' - L'$ surplus invariance reads

$$A - L \in \mathcal{A}, D_{A-L} = D_{A'-L'} \implies A' - L' \in \mathcal{A}.$$ 

Is this too strong a requirement? Shouldn’t we ask this only if $L = L'$?

$$A - L \in \mathcal{A}, D_{A-L} = D_{A'-L} \implies A' - L \in \mathcal{A}.$$ 

No, in fact: both requirements are equivalent!
A two currency world and one insurer with capital position $X_d = A_d - L_d$ expressed in original currency and $X_f = RX_d$ in foreign currency where $R$ is the exchange rate from domestic to foreign. Assume the same test is used in both currencies.

Reasonable: $X_d$ and $X_f$ should be either both acceptable or both unacceptable!

**Definition ([4])**

An acceptance set $\mathcal{A} \subset \mathbb{R}$ is said to be *numéraire invariant*, if we have

$$X \in \mathcal{A} \text{ and } R \text{ a strictly positive random variable } \implies RX \in \mathcal{A}.$$
Proposition

\( \text{VaR}_\alpha \)-acceptability is surplus and numéraire invariant

\( \rightarrow \) However, this does not invalidate the fundamental criticism of \( \text{VaR}_\alpha \):

(a) As long as \( P(X < 0) \leq \alpha \) holds it is blind to what happens on

\( \{\omega \in \Omega ; X(\omega) < 0\} \)

and, therefore, allows the build up of uncontrolled loss peaks on that set!

(b) It does not capture diversification!
Comparing ES- and VaR-acceptability (with easy numbers)

Take $\Omega := \{\omega_1, \ldots, \omega_{10}\}$ with $P(\omega_1) = \cdots = P(\omega_{10}) = \frac{1}{10}$. Assume that $X : \Omega \to \mathbb{R}$ is the capital position of an insurance company and, for simplicity, that

$$X(\omega_1) \geq \cdots \geq X(\omega_8) > X(\omega_9) > X(\omega_{10}).$$

To emulate a “Swiss Solvency Test” type environment we assume an ES test at the confidence level $\alpha = 20\%$, i.e.

$$X \text{ is acceptable } \iff - \frac{5}{1/\alpha} \left[ \frac{X(\omega_9)}{10} + \frac{X(\omega_{10})}{10} \right] = \text{ES}_{20\%}(X) \leq 0$$

To emulate a “Solvency II” type environment we assume a VaR test at the higher confidence level $\beta = 15\%$, i.e.

$$X \text{ is acceptable } \iff P(X < 0) \leq 15\%$$
ES$_\alpha$ acceptability is not surplus invariant

Two companies with capital positions $X := A - L$ and $X' := A' - L$, respectively. They have identical liabilities, different assets, but identical options to default, i.e. $D_X = D_{X'}$.

<table>
<thead>
<tr>
<th>State</th>
<th>L</th>
<th>A</th>
<th>X</th>
<th>$D_X$</th>
<th>A'</th>
<th>X'</th>
<th>$D_{X'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1, \ldots, \omega_8$</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_9$</td>
<td>12</td>
<td>13</td>
<td>1</td>
<td>0</td>
<td>15</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_{10}$</td>
<td>12</td>
<td>10</td>
<td>-2</td>
<td>2</td>
<td>10</td>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>

Under the VaR$_{15\%}$-test we have

$\rightarrow \ P(X < 0) = P(X' < 0) = 10\% \leq 15\% \implies X$ and $X'$ are both acceptable

Under ES$_{20\%}$-test we have

$\rightarrow \ ES_{20\%}(X) = -5\left[\frac{1}{10} - \frac{2}{10}\right] = \frac{1}{2} > 0 \implies X$ is not acceptable

$\rightarrow \ ES_{20\%}(X') = -5\left[\frac{3}{10} - \frac{2}{10}\right] = -\frac{1}{2} < 0 \implies X'$ is acceptable
ES$_\alpha$ acceptability is **not** numéraire invariant

One a company with capital position $X_d := A_d - L_d$, expressed in domestic currency and $X_f := RX_d$ in foreign currency where $R$ is the exchange rate from domestic to foreign. Assume the domestic and foreign regulators have either both an ES$_{20\%}$ test or both a VaR$_{15\%}$ test in their respective currencies.

<table>
<thead>
<tr>
<th>State</th>
<th>$L_d$</th>
<th>$A_d$</th>
<th>$X_d$</th>
<th>$R$</th>
<th>$X_f$</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

Under the VaR$_{15\%}$-test we have

$\rightarrow \mathbb{P}(X_d < 0) = \mathbb{P}(X_f < 0) = 10\% \leq 15\% \implies X_d$ and $X_f$ are both acceptable

Under ES$_{20\%}$-test we have

$\rightarrow$ ES$_{20\%}(X_d) = -5 \left[ \frac{1}{10} - \frac{2}{10} \right] = \frac{1}{2} > 0 \implies X_d$ is not acceptable in the domestic jurisdiction

$\rightarrow$ ES$_{20\%}(X_f) = -5 \left[ \frac{3}{10} - \frac{2}{10} \right] = -\frac{1}{2} < 0 \implies X_f$ is acceptable in the foreign jurisdiction
ES$_\alpha$ acceptability is not surplus invariant 2

Proposition ([4])

Let $X \notin \mathcal{A}^\alpha$. The following statements are equivalent:

(a) There exists $Y \in \mathcal{A}^\alpha$ such that $D_X = D_Y$;
(b) $\mathbb{P}(X < 0) < \alpha$
(c) $X \in \mathcal{A}_\beta$ for some $\beta \in (0, \alpha)$.

→ This situation arises in the region that distinguishes Solvency II (based on VaR$_{0.5\%}$) and SST (based on ES$_{1\%}$)
Are coherence, surplus invariance, and numéraire invariance compatible?

Theorem ([4])
Let $\mathcal{A}$ be a closed, coherent acceptance set. The following are equivalent:

(a) $\mathcal{A}$ is surplus invariant.
(b) $\mathcal{A}$ is numéraire invariant.
(c) $\mathcal{A}$ is of SPAN-type.

Corollary ([4])
Let $\mathcal{A}$ be a closed, convex acceptance set. The following are equivalent:

(a) $\mathcal{A}$ is numéraire invariant.
(b) $\mathcal{A}$ is of SPAN-type.

→ Unfortunately, unless $\mathcal{A} = \Omega$, acceptance sets are of the form $\text{SPAN}(\mathcal{A})$ suffer from a similar shortcoming as $\text{VaR}_\alpha$ and are blind to what happens on $\mathcal{A}^c$: They allow the build up of uncontrolled loss peaks on that set!
Conclusion

<table>
<thead>
<tr>
<th>Multiple competing requirements</th>
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<tr>
<td>Captures diversification</td>
</tr>
<tr>
<td>SPAN</td>
</tr>
<tr>
<td>VaR</td>
</tr>
<tr>
<td>ES</td>
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</table>

→ This confirms what we all know: **THE** universally ideal capital adequacy test does not exist and we need to weigh the relative importance of competing and, sometimes, mutually exclusive requirements

→ Expected Shortfall does not really take an exclusive policyholder perspective

→ A global Expected Shortfall regime would allow for regulatory arbitrage

→ The SPAN-type acceptance sets are the only coherent acceptance sets that are surplus invariant and also the only ones that are numéraire invariant

THANK YOU FOR YOUR ATTENTION!


