

# STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY PROCESSES WITH INDEPENDENT INCREMENTS

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ABSTRACT. This article considers infinite dimensional stochastic differential equations driven by processes with independent increments, which may also have fixed times of discontinuity. We are in particular interested in the existence of a càdlàg solution, which is provided by making a fixed point argument on a particular Banach space, where all processes are càdlàg. This differs from related literature in this field, where one typically works on larger spaces, in which not all processes need to be càdlàg. The stability of solutions is considered as well.

**Key Words:** infinite dimensional stochastic equations, càdlàg solutions, fixed times of discontinuity, stability of solutions

## 1. INTRODUCTION

A popular field of research in many areas of mathematics are infinite dimensional stochastic equations of the form

$$(1.1) \quad \begin{cases} d\Phi_t &= A\Phi_t dt + \sum_{i=1}^n \sigma_i(t, \Phi_{t-}) dX_t^i, \\ \Phi_0 &= h_0 \end{cases}$$

where  $A$  is the generator of a strongly continuous semigroup  $(S_t)_{t \geq 0}$  in a separable Hilbert space  $H$ . Typically, one is interested in a so-called mild solution to (1.1), which satisfies

$$(1.2) \quad \Phi_t = S_t h_0 + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Phi_{s-}) dX_s^i, \quad t \in \mathbb{R}_+.$$

In our framework, we allow time-inhomogeneous processes with independent increments as drivers  $X = (X^1, \dots, X^n)$ . They may also have fixed times of discontinuity. In particular,  $X^1$  can be the running time  $t$  and  $X^2$  a Lévy process.

Equations of the type (1.1) have a wide field of applications, e.g. in physics and economics. The main reference for stochastic differential equations of the form (1.1), where  $X$  is an infinite dimensional Wiener process, is Da Prato and Zabczyk [2].

For several reasons, there has recently been a growing interest in stochastic equations of the type (1.1) with jump noise terms  $X$ . We mention [9, 10, 8, 19, 22] and the forthcoming textbook [21], where  $X$  is a Lévy process, and [1, 16, 17, 11, 12], where one has a compensated Poisson random measure as driving term.

In applications, one is often not only interested in the existence of a unique solution to (1.1), the solution should also have regular sample paths, i.e. it should be càdlàg or continuous, respectively, provided the equation is driven by a continuous process.

Many authors, which deal with path properties of the solution, proceed as follows: First, they show, using the Banach fixed point theorem, the existence of a solution to (1.1) on a large space of processes which also covers processes that are not càdlàg. The typical norm of such a Banach space is

$$(1.3) \quad \|\Phi\| = \left( \sup_{t \in [0, T]} \mathbb{E} [\|\Phi_t\|_H^2] \right)^{\frac{1}{2}},$$

where  $T > 0$  is a finite time horizon, see e.g. [10, 17].

In a second step, they show that the stochastic convolutions, appearing on the right-hand side of (1.2), have a càdlàg modification, implying that the solution  $\Phi$  has a càdlàg modification. So it is done, e.g. in [2] for the continuous case and in [8] for the Lévy case.

In [2], the existence of a continuous modification is shown by using Yosida approximations of the generator  $A$ . Other typical tools are the Kotelenetz theorem (see [18]) or the Szekőfalvi-Nagy's theorem on unitary dilations (see e.g. [25, Thm. I.8.1], or [3, Sec. 7.2]), which has been applied for establishing the regularity of infinite dimensional processes for instance in [14] and [13].

In this paper we present a direct approach to achieve the existence of a càdlàg solution to (1.1) by making a fixed point argument on a smaller Banach space (the space  $S^2[0, T]$ , see Section 3) consisting only of càdlàg processes and equipped with the norm

$$\|\Phi\| = \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|_H^2 \right]^{\frac{1}{2}},$$

which is stronger than (1.3).

The remainder of this text is organized as follows. Sections 2–4 provide the required foundations for dealing with Hilbert space valued stochastic equations: In Section 2 we present the driving processes, Section 3 contains results about Hilbert space valued processes, in particular we introduce the space  $S^2[0, T]$ , and in Section 4 we briefly discuss the stochastic integral in infinite dimension and provide some properties that will be used subsequently. Section 5 contains our main result, Theorem 5.5, concerning the existence of a unique càdlàg solution to Hilbert space valued stochastic equations. In Section 6, we deal with the stability of such equations. Section 7 concludes.

## 2. THE DRIVING PROCESSES

Throughout this text,  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  denotes a filtered probability space satisfying the usual conditions. The predictable  $\sigma$ -algebra is denoted by  $\mathcal{P}$ .

As driving processes for stochastic differential equations, we shall consider real-valued special semimartingales  $X$  with canonical decomposition  $X = B + M^c + M^d$ , where  $B$  is of finite variation on compact intervals,  $M^c$  is a continuous martingale and  $M^d$  a purely discontinuous martingale (see [15, Def. I.4.11]). We suppose that:

- $B : \mathbb{R}_+ \rightarrow \mathbb{R}$  is deterministic and continuous.
- $B_0 = 0$ ,  $M_0^c = 0$ ,  $M_0^d = 0$  ( $\mathbb{P}$ -a.s.) and  $\mathbb{E}[(M_t^c)^2] + \mathbb{E}[(M_t^d)^2] < \infty$  for each  $t \in \mathbb{R}_+$ .
- The predictable quadratic covariation (see [15, Thm. I.4.2])  $C = \langle M^c, M^c \rangle : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is deterministic and continuous.
- The compensator  $\nu(dt, dx)$  of the random measure  $\mu^{M^d}$  of jumps (see [15, Thm. II.1.8]) is deterministic.

Note that, in this case, the characteristics  $(B, C, \nu)$  of  $X$  are deterministic, and hence  $X$  is a process with independent increments by [15, Thm. II.4.15].

We denote by SPII the set of all of special semimartingales satisfying the conditions above. Let us, for the rest of this section, investigate the properties of these processes, which are required in the sequel. Let  $X = M + B \in \text{SPII}$ , where  $M$  denotes the martingale  $M = M^c + M^d$ .

A time  $t \in \mathbb{R}_+$  is called a *fixed time of discontinuity* for  $X$  if  $\mathbb{P}(\Delta X_t \neq 0) > 0$ . According to [15, Thm. II.4.15], for  $X \in \text{SPII}$  a time  $t \in \mathbb{R}_+$  is a fixed time of discontinuity for  $X$  if and only if  $\nu(\{t\} \times \mathbb{R}) > 0$ . In this case, we have  $\nu(\{t\} \times \mathbb{R}) \leq 1$  and the distribution of the random variable  $\Delta X_t$  is

$$\nu(\{t\} \times dx) + [1 - \nu(\{t\} \times \mathbb{R})]\delta_0(dx),$$

where  $\delta_0(dx)$  denotes the Dirac measure in zero.

We introduce the non-decreasing function  $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  as

$$D_t := \int_0^t \int_{\mathbb{R}} x^2 \nu(ds, dx).$$

Notice that  $D_t = \mathbb{E}[\sum_{s \leq t} (\Delta X_s)^2] = \mathbb{E}[[M^d, M^d]_t]$ , which is finite by [15, Prop. 4.50.c], whence  $D$  is well-defined. The function  $D$  is càdlàg and it has a jump at  $t \in \mathbb{R}_+$  if and only if  $X$  has a fixed time of discontinuity at  $t$ . In this case, we have

$$\Delta D_t = \mathbb{E}[\Delta X_t^2] = \int_{\mathbb{R}} x^2 \nu(\{t\}, dx).$$

To each  $X \in \text{SPII}$ , we associate the *variation function*  $v_X : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,

$$(2.1) \quad v_X(t) := \text{Var}(B)_t + 2\sqrt{C_t + D_t}, \quad t \in \mathbb{R}_+.$$

This definition is motivated by the forthcoming estimate (5.4), see Proposition 5.2.

Note that every Lévy process  $X$  with  $\mathbb{E}[X_1^2] < \infty$  belongs to SPII. More generally, every square-integrable PIIAC – also called non-homogeneous Lévy process – in the sense of [4], [5] and [6], which is a process with independent increments whose characteristics are absolutely continuous in time, belongs to SPII. These are examples of processes with independent increments with no fixed times of discontinuity and, therefore, continuous variation functions.

For a vector  $X = (X^1, \dots, X^n)$  with  $X^i \in \text{SPII}$  for  $i = 1, \dots, n$  we call  $t \in \mathbb{R}_+$  a *fixed time of discontinuity* for  $X$  if there is an index  $i \in \{1, \dots, n\}$  such that  $t$  is a fixed time of discontinuity for  $X^i$ , and we define the *variation function*  $v_X$  as

$$v_X(t) := \sum_{i=1}^n v_{X^i}(t), \quad t \in \mathbb{R}_+.$$

Note that by the arguments above  $v_X$  starts in zero, is non-decreasing and càdlàg, and that it has a jump at  $t \in \mathbb{R}_+$  if and only if  $X$  has a fixed time of discontinuity at  $t$ .

We say that  $X$  has *no accumulating fixed times of discontinuity*, if there exists a constant  $\delta > 0$  such that for all  $s, t \in \mathbb{R}_+$  with  $s \neq t$  the relation  $\Delta v_X(s), \Delta v_X(t) > 0$  implies  $|s - t| \geq \delta$ .

The following result will be useful for the establishment of some results regarding stochastic integration later, cf. Lemma 4.1 and Proposition 5.2.

**2.1. Lemma.** *For every real-valued predictable process  $Y$  we have*

$$\mathbb{E} \left[ \int_0^t Y_s^2 d\langle M, M \rangle_s \right] = \mathbb{E} \left[ \int_0^t Y_s^2 d(C + D)_s \right], \quad t \in \mathbb{R}_+.$$

*Proof.* By [15, Prop. I.4.50.b],  $\langle M, M \rangle$  is the compensator of  $[M, M]$ , whence

$$(2.2) \quad \mathbb{E} \left[ \int_0^t Y_s^2 d\langle M, M \rangle_s \right] = \mathbb{E} \left[ \int_0^t Y_s^2 d[M, M]_s \right].$$

According to [15, Thm. I.4.52], it holds

$$(2.3) \quad [M, M]_t = \langle M^c, M^c \rangle_t + \sum_{s \leq t} \Delta M_s^2.$$

The claimed identity follows from (2.2) and (2.3).  $\square$

In order to prove the existence of solutions of stochastic differential equations in Section 5, the following auxiliary result will be needed.

**2.2. Lemma.** *Let  $X \in \text{SPII}$  and  $T > 0$  be no fixed time of discontinuity for  $X$ . Then we have  $X^T, X - X^T \in \text{SPII}$  with variation functions*

$$(2.4) \quad v_{X^T}(t) = \begin{cases} v_X(t), & t \in [0, T) \\ v_X(T), & t \in [T, \infty) \end{cases}$$

and

$$(2.5) \quad v_{X - X^T}(t) = \begin{cases} 0, & t \in [0, T) \\ v_X(t) - v_X(T), & t \in [T, \infty). \end{cases}$$

*Proof.* The finite variation parts  $B^T$  and  $B - B^T$  are again deterministic and continuous, and we have  $\text{Var}(B^T) = \text{Var}(B)^T$  as well as  $\text{Var}(B - B^T) = \text{Var}(B) - \text{Var}(B)^T$ .

For the continuous martingale parts  $(M^c)^T$  and  $M^c - (M^c)^T$  we obtain, by noting that  $\langle N, N \rangle = [N, N]$  for every continuous martingale  $N$ ,

$$\langle M^T, M^T \rangle = \langle M, M \rangle^T$$

and, by using  $\langle M, M^T \rangle = \langle M, M \rangle^T$  and the bilinearity of the predictable quadratic covariation,

$$\langle M - M^T, M - M^T \rangle = \langle M, M \rangle - 2\langle M, M^T \rangle + \langle M^T, M^T \rangle = \langle M, M \rangle - \langle M, M \rangle^T.$$

We define the deterministic measures  $\nu^1(dt, dx)$  and  $\nu^2(dt, dx)$  as

$$\begin{aligned} \nu_1(A \times B) &:= \nu(A \cap [0, T] \times B), \\ \nu_2(A \times B) &:= \nu(A \cap (T, \infty) \times B) \end{aligned}$$

for  $A \in \mathcal{B}(\mathbb{R}_+)$  and  $B \in \mathcal{B}(\mathbb{R})$ . Then, we get for every  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable function  $W$

$$\begin{aligned}\mathbb{E}[W * \mu_\infty^{X^T}] &= \mathbb{E}[W \mathbb{1}_{[0,T]} * \mu_\infty^X] = \mathbb{E}[W \mathbb{1}_{[0,T]} * \nu_\infty] = \mathbb{E}[W * \nu_\infty^1], \\ \mathbb{E}[W * \mu_\infty^{X-X^T}] &= \mathbb{E}[W \mathbb{1}_{(T,\infty)} * \mu_\infty^X] = \mathbb{E}[W \mathbb{1}_{(T,\infty)} * \nu_\infty] = \mathbb{E}[W * \nu_\infty^2],\end{aligned}$$

because  $W \mathbb{1}_{[0,T]}$  and  $W \mathbb{1}_{(T,\infty)}$  are also  $\mathcal{P} \otimes \mathcal{B}$ -measurable, see [15, Thm. II.1.8]. Thus,  $\nu^1$  and  $\nu^2$  are the compensators of  $X^T$  and  $X - X^T$ .

Consequently, the characteristics of  $X^T$  and  $X - X^T$  are deterministic, implying that  $X^T, X - X^T \in \text{SPII}$  by [15, Thm. II.4.15], and the variation functions are of the claimed forms (2.4) and (2.5).  $\square$

### 3. HILBERT SPACE VALUED PROCESSES

Let  $H$  denote a separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and associated norm  $\|\cdot\|$ . We fix a finite time horizon  $T > 0$ .

An adapted  $H$ -valued process  $(\Phi_t)_{t \in [0,T]}$  is called a *martingale* if

- $\mathbb{E}[\|\Phi_t\|] < \infty$  for all  $t \in [0, T]$ ;
- $\mathbb{E}[\Phi_t | \mathcal{F}_s] = \Phi_s$  ( $\mathbb{P}$ -a.s.) for all  $0 \leq s \leq t \leq T$ .

For the notion of conditional expectation of random variables having values in a separable Banach space, we refer to [2, Sec. 1.3].

An indispensable tool will be *Doob's martingale inequality*

$$(3.1) \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \|\Phi_t\|^2 \right] \leq 4 \sup_{t \in [0,T]} \mathbb{E} [\|\Phi_t\|^2] = 4\mathbb{E} [\|\Phi_T\|^2],$$

valid for every  $H$ -valued càdlàg martingale  $\Phi$ , which is a consequence of Thm. 3.8 and Prop. 3.7 in [2].

Let  $S^2[0, T]$  be the linear space of all adapted càdlàg  $H$ -valued processes such that

$$\|\Phi\|_{S^2[0,T]} := \mathbb{E} \left[ \sup_{t \in [0,T]} \|\Phi_t\|^2 \right]^{\frac{1}{2}} < \infty.$$

Note that  $\|\cdot\|_{S^2[0,T]}$  defines a norm on the linear space  $S^2[0, T]$ . Since, in Section 5, we will tackle the existence of solutions to stochastic equations by making a fixed point argument on  $S^2[0, T]$ , the following result will be crucial.

**3.1. Proposition.** *The normed space  $(S^2[0, T], \|\cdot\|_{S^2[0,T]})$  is a Banach space.*

*Proof.* Let  $(\Phi^n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $S^2[0, T]$ . Since it suffices to show that a subsequence of  $(\Phi^n)$  converges in  $S^2[0, T]$ , we may, without loss of generality, assume that

$$(3.2) \quad \|\Phi^{n+1} - \Phi^n\|_{S^2[0,T]} \leq \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}.$$

Using the Markov inequality and (3.2), we have

$$\mathbb{P} \left( \sup_{t \in [0,T]} \|\Phi_t^{n+1} - \Phi_t^n\| \geq \frac{1}{2^n} \right) \leq \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}.$$

The Borel-Cantelli lemma implies that for almost all  $\omega \in \Omega$  the sequence  $(\Phi^n(\omega))$  is a Cauchy sequence in the space  $\mathbb{D}$  of càdlàg functions mapping from  $[0, T]$  to  $H$ , equipped with the supremum-norm. By the completeness of  $H$ , the normed space  $\mathbb{D}$  is a Banach space. Consequently there is an adapted, càdlàg process  $\Phi : \Omega \times [0, T] \rightarrow H$  such that

$$\sup_{t \in [0, T]} \|\Phi_t - \Phi_t^n\| \rightarrow 0 \quad \mathbb{P}\text{-a.s. as } n \rightarrow \infty.$$

By the continuity of the supremum-norm in  $\mathbb{D}$ , the monotone convergence theorem and (3.2), we obtain

$$\begin{aligned} \|\Phi - \Phi^n\|_{S^2[0, T]} &= \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t - \Phi_t^n\|^2 \right]^{\frac{1}{2}} = \mathbb{E} \left[ \lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \left\| \sum_{k=n}^m (\Phi_t^{k+1} - \Phi_t^k) \right\|^2 \right]^{\frac{1}{2}} \\ &\leq \mathbb{E} \left[ \lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \left( \sum_{k=n}^m \|\Phi_t^{k+1} - \Phi_t^k\| \right)^2 \right]^{\frac{1}{2}} = \lim_{m \rightarrow \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left( \sum_{k=n}^m \|\Phi_t^{k+1} - \Phi_t^k\| \right)^2 \right]^{\frac{1}{2}} \\ &\leq \lim_{m \rightarrow \infty} \left\| \sum_{k=n}^m \sup_{t \in [0, T]} \|\Phi_t^{k+1} - \Phi_t^k\| \right\|_{L^2(\Omega)} \leq \lim_{m \rightarrow \infty} \sum_{k=n}^m \|\Phi^{k+1} - \Phi^k\|_{S^2} \leq \frac{1}{2^{n-1}}. \end{aligned}$$

We deduce that  $\|\Phi\|_{S^2[0, T]} < \infty$  and  $\Phi^n \rightarrow \Phi$  in  $S^2[0, T]$ .  $\square$

#### 4. STOCHASTIC INTEGRATION

For a predictable  $H$ -valued process  $\Phi$  and a real-valued semimartingale  $X$  we can define the stochastic integral  $\int_0^t \Phi_s dX_s$ . The construction is just as for real-valued integrands, see e.g. in Jacod and Shiryaev [15] or Protter [23]. We shall briefly sketch the construction in the case where  $X = M + B \in \text{SPII}$  (see Section 2) is a process with independent increments.

For a predictable process  $\Phi$  with  $\sup_{t \in [0, T]} \|\Phi_t\| < \infty$  ( $\mathbb{P}$ -a.s.) we define  $\int_0^t \Phi_s dB_s$  pathwise as a Bochner integral. The  $H$ -valued integral process is adapted, càdlàg and has paths of finite variation.

For a predictable process  $\Phi \in L^2(M) := L^2(\Omega \times [0, T], \mathcal{P}, \mathbb{P} \otimes \langle M, M \rangle; H)$  we define  $\int_0^t \Phi dM_s$  first for simple integrands and then extend it via the Itô-isometry

$$(4.1) \quad \mathbb{E} \left[ \left\| \int_0^t \Phi_s dM_s \right\|^2 \right] = \mathbb{E} \left[ \int_0^t \|\Phi_s\|^2 d\langle M, M \rangle_s \right],$$

which is then also valid for every  $\Phi \in L^2(M)$ . The integral process is an  $H$ -valued martingale with càdlàg paths. In order to get the Itô-isometry (4.1), it is vital that the state space  $H$  is a Hilbert space.

Finally, for a predictable process  $\Phi \in L^2(M)$  satisfying  $\sup_{t \in [0, T]} \|\Phi_s\| < \infty$  ( $\mathbb{P}$ -a.s.) we set

$$\int_0^t \Phi_s dX_s := \int_0^t \Phi_s dB_s + \int_0^t \Phi_s dM_s.$$

The stochastic integral is bilinear with respect to  $(\Phi, X)$ , and for every continuous linear operator  $A \in \mathcal{L}(H)$  we have

$$(4.2) \quad \int_0^t A\Phi_s dX_s = A \int_0^t \Phi_s dX_s.$$

The construction of the stochastic integral in the more general situation, where the driving semimartingale may also be infinite dimensional, can be found in Métivier [20].

We remark that the stochastic integral can also be defined on appropriate Banach spaces, so-called M-type 2 spaces. Then the integral is still a bounded linear operator, but no isometry, in general. We refer to [24] for further details.

The following lemma shows that for each  $\Phi \in S^2[0, T]$  the stochastic integral  $\int_0^t \Phi_{s-} dX_s$  exists.

**4.1. Lemma.** *For every  $\Phi \in S^2[0, T]$ , the process  $\Phi_-$  is predictable, we have  $\Phi_- \in L^2(M)$  and  $\sup_{t \in [0, T]} \|\Phi_{t-}\| < \infty$  ( $\mathbb{P}$ -a.s.).*

*Proof.* Since  $\Phi_-$  is left-continuous with right-hand limits, it is predictable and we have  $\sup_{t \in [0, T]} \|\Phi_{t-}\| < \infty$  ( $\mathbb{P}$ -a.s.). Using Lemma 2.1, we obtain

$$\mathbb{E} \left[ \int_0^T \|\Phi_{s-}\|^2 d\langle M, M \rangle_s \right] \leq (C_T + D_T) \|\Phi\|_{S^2}^2 < \infty,$$

that is,  $\Phi_- \in L^2(M)$ . □

## 5. STOCHASTIC DIFFERENTIAL EQUATIONS

Now let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup in the separable Hilbert space  $H$ , i.e. a family of bounded linear operators  $S_t : H \rightarrow H$  such that

- $S_0 = \text{Id}$ ;
- $S_{s+t} = S_s S_t$  for all  $s, t \geq 0$ ;
- $\lim_{t \rightarrow 0} S_t h = h$  for all  $h \in H$ ;

with generator  $A : \mathcal{D}(A) \subset H \rightarrow H$ . By  $\|\cdot\|_{\mathcal{L}(H)}$  we denote the operator norm of a bounded linear operator. The semigroup  $(S_t)$  is called *contractive on  $H$*  if

$$\|S_t\|_{\mathcal{L}(H)} \leq 1, \quad t \geq 0$$

and *pseudo-contractive on  $H$*  if there is a constant  $\omega \geq 0$  such that

$$\|S_t\|_{\mathcal{L}(H)} \leq e^{\omega t}, \quad t \geq 0.$$

In this section, we intend to find mild solutions of stochastic differential equations of the type (1.1) driven by processes  $X^1, \dots, X^n \in \text{SPII}$  (see Section 2) with independent increments, for each initial condition  $h_0 \in H$ . That is, a process  $(\Phi_t)_{t \geq 0}$  satisfying (1.2). We also intend to establish the existence of a weak solution  $(\Phi_t)_{t \geq 0}$  to (1.1), i.e.  $(\Phi_t)$  satisfies, for all  $\zeta \in \mathcal{D}(A^*)$ ,

$$(5.1) \quad \langle \zeta, \Phi_t \rangle = \langle \zeta, h_0 \rangle + \int_0^t \langle A^* \zeta, \Phi_s \rangle ds + \sum_{i=1}^n \int_0^t \langle \zeta, \sigma_i(s, \Phi_{s-}) \rangle dX_s^i, \quad t \in \mathbb{R}_+.$$

We shall now, step by step, approach our main result, Theorem 5.5. Our first auxiliary result, Lemma 5.1, shows that the volatilities on the right-hand side of (1.2) are integrable for  $\Phi \in S^2[0, T]$ , that is, the stochastic convolutions are well-defined. Then, Proposition 5.2 shows that these convolutions actually belong again to  $S^2[0, T]$  and the estimate (5.4) is provided.

If the interval  $[0, T]$  is small enough, we immediately get the existence of a unique solution in  $S^2[0, T]$  from the Banach fixed point theorem, see Lemma 5.3. Afterwards, Lemma 5.4 tells us how to extend such a solution to a larger interval.

In what follows, let  $H_0 \subset H$  be a closed subspace such that  $(S_t)$  is pseudo-contractive on  $H_0$  with constant  $\omega \geq 0$ . We denote by  $X = (X^1, \dots, X^n)$  the vector consisting of the driving processes.

**5.1. Lemma.** *Let  $\sigma : \mathbb{R}_+ \times H \rightarrow H$  be continuous. Assume there is a constant  $L \geq 0$  such that*

$$(5.2) \quad \|\sigma(t, h_1) - \sigma(t, h_2)\| \leq L\|h_1 - h_2\|$$

for all  $t \in \mathbb{R}_+$  and all  $h_1, h_2 \in H$ . Then, for each  $\Phi \in S^2[0, T]$ , the process  $\Psi_t := \sigma(t, \Phi_t)$  belongs to  $S^2[0, T]$ .

*Proof.* The process  $\Psi$  is adapted and càdlàg. It remains to show  $\|\Psi\|_{S^2[0, T]} < \infty$ . By the continuity of  $\sigma$  there is a constant  $C_T > 0$  such that  $\|\sigma(t, 0)\| \leq C_T$  for all  $t \in [0, T]$ . Therefore, we get for all  $t \in [0, T]$  and  $h \in H$ , by using (5.2), the inequality

$$(5.3) \quad \|\sigma(t, h)\| \leq \|\sigma(t, 0)\| + \|\sigma(t, h) - \sigma(t, 0)\| \leq (L \wedge C_T)(1 + \|h\|).$$

Taking into account that we have for each  $\Phi \in S^2[0, T]$

$$\mathbb{E} \left[ \sup_{t \in [0, T]} (1 + \|\Phi_t\|)^2 \right] \leq 2 + 2\mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right] < \infty,$$

we deduce  $\|\Psi\|_{S^2[0, T]} < \infty$  from (5.3).  $\square$

**5.2. Proposition.** *Let  $T > 0$  and  $\Phi \in S^2[0, T]$  be such that  $\Phi_t \in H_0$  for all  $t \in [0, T]$ . Let  $S \in [0, T]$  and  $X \in \text{SPII}$  be such that  $X_t = 0$ ,  $t \in [0, S]$ . Then  $\Psi_t := \int_0^t S_{t-s} \Phi_s dX_s$  belongs to  $S^2[0, T]$  and the following estimate is valid:*

$$(5.4) \quad \|\Psi\|_{S^2[0, T]} \leq e^{\omega(T-S)} v_X(T) \|\Phi\|_{S^2[0, T]}.$$

*Proof.* First, we note that  $\Psi$  is adapted. The  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  defined as

$$(5.5) \quad T_t := e^{-\omega t} S_t, \quad t \in \mathbb{R}_+$$

is contractive on  $H_0$ . By the Szekőfalvi-Nagy's theorem on unitary dilations (see e.g. [25, Thm. I.8.1], or [3, Sec. 7.2]), there exists another separable Hilbert space  $\mathcal{H}_0$  and a strongly continuous unitary group  $(U_t)_{t \in \mathbb{R}}$  in  $\mathcal{H}_0$  such that the diagram

$$\begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{U_t} & \mathcal{H}_0 \\ \uparrow \ell & & \downarrow \pi \\ H_0 & \xrightarrow{T_t} & H_0 \end{array}$$



commutes for every  $t \in \mathbb{R}_+$ , where  $\ell : H_0 \rightarrow \mathcal{H}_0$  is an isometric embedding (hence the adjoint operator  $\pi := \ell^*$  is the orthogonal projection from  $\mathcal{H}_0$  into  $H_0$ ), that is

$$(5.6) \quad \pi U_t \ell h = T_t h \quad \text{for all } t \in \mathbb{R}_+ \text{ and } h \in H_0.$$

We obtain for all  $t \geq 0$  by using equation (5.6) and (4.2)

$$(5.7) \quad \begin{aligned} \Psi_t &= \int_0^t S_{t-s} \Phi_s - dX_s = \int_0^t e^{\omega(t-s)} T_{t-s} \Phi_s - dX_s \\ &= e^{\omega t} \int_0^t e^{-\omega s} \pi U_{t-s} \ell \Phi_s - dX_s = e^{\omega t} \pi U_t \int_0^t e^{-\omega s} U_{-s} \ell \Phi_s - dX_s. \end{aligned}$$

Note that  $e^{-\omega s} U_{-s} \ell \Phi_s$  belongs to  $S^2[0, T]$ . The stochastic integral  $\int_0^t e^{-\omega s} U_{-s} \ell \Phi_s - dX_s$  appearing on the right-hand side of equation (5.7) is càdlàg. Therefore, the process  $\Psi$  is càdlàg, too, because  $(t, h) \mapsto U_t h$  is uniformly continuous on compact subsets, see e.g. [7, Lemma I.5.2].

It remains to prove the inequality (5.4), which then in particular implies  $\Psi \in S^2[0, T]$ . Let  $X = M + B$  be the canonical decomposition of  $X$ . We obtain, noting that  $\|\ell\|_{\mathcal{L}(H_0; \mathcal{H}_0)} = \|\pi\|_{\mathcal{L}(\mathcal{H}_0; H_0)} = 1$ ,  $\|T_t\|_{\mathcal{L}(H_0)} \leq 1$ ,  $t \in \mathbb{R}_+$  and  $\|U_t\|_{\mathcal{L}(\mathcal{H}_0)} \leq 1$ ,  $t \in \mathbb{R}$ , and by using equation (5.7), Hölder's inequality and  $B_t = 0$ ,  $t \in [0, S]$  by hypothesis,

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t S_{t-s} \Phi_s - dB_s \right\|^2 \right] &\leq \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| e^{\omega(t-S)} \pi U_t \int_0^t e^{-\omega(s-S)} U_{-s} \ell \Phi_s - dB_s \right\|^2 \right] \\ &\leq e^{2\omega(T-S)} \mathbb{E} \left[ \sup_{t \in [0, T]} \left( \int_0^t \|e^{-\omega(s-S)} U_{-s} \ell \Phi_s -\| d\text{Var}(B)_s \right)^2 \right] \\ &\leq e^{2\omega(T-S)} \text{Var}(B)_T^2 \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right]. \end{aligned}$$

Using equation (5.7), Doob's martingale inequality (3.1), the Itô isometry (4.1), Lemma 2.1 and  $M_t = 0$ ,  $t \in [0, S]$  by hypothesis, we obtain

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t S_{t-s} \Phi_s - dM_s \right\|^2 \right] &\leq \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| e^{\omega(t-S)} \pi U_t \int_0^t e^{-\omega(s-S)} U_{-s} \ell \Phi_s - dM_s \right\|^2 \right] \\ &\leq 4e^{2\omega(T-S)} \mathbb{E} \left[ \left\| \int_0^T e^{-\omega(s-S)} U_{-s} \ell \Phi_s - dM_s \right\|^2 \right] \leq 4e^{2\omega(T-S)} \mathbb{E} \left[ \int_0^T \|\Phi_s -\|^2 d\langle M, M \rangle_s \right] \\ &\leq 4e^{2\omega(T-S)} (C_T + D_T) \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right], \end{aligned}$$

establishing the estimate (5.4) by the definition (2.1) of the variation function  $v_X$ .  $\square$

Proposition 5.2 shows why we demand pseudo-contractivity of  $(S_t)$  in a closed subspace. Namely, for the proof of the preceding Proposition 5.2, we have followed the idea of Hausenblas and Seidler [14], using the Szekőfalvi-Nagy's theorem on unitary dilations. In order to get the existence of a càdlàg modification of the stochastic convolution  $\Psi$ , we could also directly use the Kotelenez theorem (see Kotelenez [18]). Note that the Kotelenez theorem also requires pseudo-contractivity of the semigroup.

With the unitary dilation we get another, rather simple proof of this known result. For stochastic convolutions driven by compensated Poisson random measures, a proof of the càdlàg property, again provided the semigroup is pseudo-contractive, can be found in [1, Prop. 2.4].

In the next result, Lemma 5.3, we shall consider integral equations of the type (5.10) with a generalized initial condition, which are obviously an extension of (1.2).

**5.3. Lemma.** *Let  $\sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H_0$  be continuous. Let  $T > 0$ ,  $S \in [0, T]$  and  $X$  be such that  $X_t = 0$ ,  $t \in [0, S]$ . Assume there is a constant  $L \geq 0$  such that*

$$(5.8) \quad Le^{\omega(T-S)}v_X(T) < 1,$$

$$(5.9) \quad \|\sigma_i(t, h_1) - \sigma_i(t, h_2)\| \leq L\|h_1 - h_2\|, \quad i = 1, \dots, n$$

for all  $t \in \mathbb{R}_+$  and all  $h_1, h_2 \in H$ . Then, for each  $\Xi \in S^2[0, T]$ , there exists a unique solution  $\Phi \in S^2[0, T]$  for

$$(5.10) \quad \Phi_t = \Xi_t + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Phi_{s-}) dX_s^i.$$

*Proof.* For each  $\Phi \in S^2[0, T]$ , we set

$$\Lambda(\Phi)_t := \Xi_t + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Phi_{s-}) dX_s^i, \quad t \in [0, T]$$

which is well-defined by Lemma 5.1, and belongs to  $S^2[0, T]$  according to Proposition 5.2. Using the estimate (5.4) from Proposition 5.2 and assumption (5.9), we obtain

$$\|\Lambda(\Phi^1) - \Lambda(\Phi^2)\|_{S^2[0, T]} \leq Le^{\omega(T-S)}v_X(T)\|\Phi^1 - \Phi^2\|_{S^2[0, T]}, \quad \Phi^1, \Phi^2 \in S^2[0, T].$$

Since  $S^2[0, T]$  is a Banach space according to Proposition 3.1 and  $\Lambda : S^2[0, T] \rightarrow S^2[0, T]$  a contraction by (5.8), we apply the Banach fixed point theorem to the equation  $\Lambda(\Phi) = \Phi$ , which gives us the existence of a unique solution  $\Phi \in S^2[0, T]$  to (5.10).  $\square$

**5.4. Lemma.** *Let  $0 < S < T$  be such that  $S$  is no fixed time of discontinuity for  $X$ . Let  $\Xi \in S^2[0, T]$  and  $\sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H_0$  be continuous. Assume there is a constant  $L \geq 0$  satisfying*

$$(5.11) \quad Le^{\omega(T-S)}(v_X(T) - v_X(S)) < 1$$

and (5.9). Provided, there exists a unique solution  $\Psi \in S^2[0, S]$  to (5.10), there is also a unique solution  $\Phi \in S^2[0, T]$  to (5.10).

*Proof.* By Lemma 2.2, we have  $(X^i)^S, X^i - (X^i)^S \in \text{SPII}$  for all  $i = 1, \dots, n$ . By hypothesis, there exists a unique solution  $\Psi \in S^2[0, S]$  to (5.10). Note that  $\Psi$  also fulfills

$$\Psi_t = \Xi_t + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Psi_{s-}) d(X^i)_s^S, \quad t \in [0, S].$$

Since the variation function of  $X^i - (X^i)^S$  is given by (2.5), we deduce from Lemma 5.3, by taking into account (5.11), the existence of a unique solution  $\Phi \in S^2[0, T]$  for

$$\Phi_t = \Xi_t + \sum_{i=1}^n \int_0^{t \wedge S} S_{t-s} \sigma_i(s, \Psi_{s-}) d(X^i)_s^S + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Phi_{s-}) d(X^i - (X^i)^S)_s$$

on  $[0, T]$ . Note that  $\Phi = \Psi$  on  $[0, S]$ , because  $X - X^S = 0$  on  $[0, S]$ , and therefore  $\Phi$  also solves (5.10).

It remains to prove uniqueness of the solution. Let  $\Phi^1, \Phi^2 \in S^2[0, T]$  be two solutions of (5.10). Since, by hypothesis, there is a unique solution  $\Psi \in S^2[0, S]$  to (5.10) on  $[0, S]$ , we conclude  $\Phi^1 = \Phi^2 = \Psi$  on  $[0, S]$ . This yields

$$\begin{aligned} \Phi_t^1 &= \Xi_t + \sum_{i=1}^n \int_0^{t \wedge S} S_{t-s} \sigma_i(s, \Psi_{s-}) d(X^i)_s^S + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Phi_{s-}^1) d(X^i - (X^i)^S)_s, \\ \Phi_t^2 &= \Xi_t + \sum_{i=1}^n \int_0^{t \wedge S} S_{t-s} \sigma_i(s, \Psi_{s-}) d(X^i)_s^S + \sum_{i=1}^n \int_0^t S_{t-s} \sigma_i(s, \Phi_{s-}^2) d(X^i - (X^i)^S)_s \end{aligned}$$

on  $[0, T]$ , implying  $\Phi^1 = \Phi^2$  on  $[0, T]$  by Lemma 5.3.  $\square$

After the preceding preparations, here is our main existence and uniqueness result:

**5.5. Theorem.** *Let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup in  $H$ , and  $H_0 \subset H$  be a closed subspace such that  $(S_t)$  is pseudo-contractive on  $H_0$ . Let  $\sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H_0$  be continuous. Assume there is a constant  $L \geq 0$  such that (5.9) is valid for all  $t \in \mathbb{R}_+$  and all  $h_1, h_2 \in H$ . Furthermore, we assume that  $X$  has no accumulating fixed times of discontinuity and*

$$(5.12) \quad L \Delta v_X(t) < 1, \quad t \in \mathbb{R}_+.$$

*Then, for each  $h_0 \in H$ , there exists a unique mild and a unique weak adapted càdlàg solution  $(\Phi_t)_{t \geq 0}$  to (1.1) with  $\Phi_0 = h_0$  satisfying*

$$(5.13) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t\|^2 \right] < \infty \quad \text{for all } T > 0.$$

*Proof.* It suffices to show existence and uniqueness on  $[0, T]$  for every  $T > 0$  which is no fixed time of discontinuity for  $X$ .

Since  $X$  has no accumulating fixed times of discontinuity and by condition (5.12), there exists a decomposition  $0 = T_0 < T_1 < \dots < T_k = T$ , consisting only of no fixed times of discontinuity for  $X$ , such that

$$(5.14) \quad L e^{\omega(T_{j+1}-T_j)} (v_X(T_{j+1}) - v_X(T_j)) < 1, \quad j = 0, \dots, k-1.$$

Note that  $\Xi_t := S_t h_0$  belongs to  $S^2[0, T]$ . Using (5.14) for  $j = 0$ , there exists, according to Lemma 5.3, a unique solution  $\Phi^0 \in S^2[0, T_1]$  to (1.2).

Inductively, using relation (5.14) and Lemma 5.4, we obtain the existence of a unique solution  $\Phi^j \in S^2[0, T_{j+1}]$  to (1.2) for  $j = 1, \dots, k-1$ . So, we finally arrive at the existence of a unique solution  $\Phi \in S^2[0, T]$  for equation (1.2). This establishes the existence of a unique mild adapted càdlàg solution to (1.1), which satisfies (5.13), because  $\Phi \in S^2[0, T]$ .

We can now proceed as in the proof of [8, Thm. C.1] to show that  $\Phi$  satisfies (5.1) for all  $\zeta \in \mathcal{D}(A^*)$  and that it is the unique weak solution to (1.1).  $\square$

Condition (5.12) only concerns fixed times of discontinuity for  $X$ . In particular, every  $X \in \text{SPII}$  having no fixed times of discontinuity, e.g. every square-integrable Lévy process, satisfies the conditions of Theorem 5.5.

In the special situation where  $A \in \mathcal{L}(H)$ , i.e.  $A$  is a bounded linear operator, we can now easily establish the existence of a strong solution  $(\Phi_t)_{t \geq 0}$  to (1.1), that is we have

$$(5.15) \quad \Phi_t = h_0 + \int_0^t A\Phi_s ds + \sum_{i=1}^n \int_0^t \sigma_i(s, \Phi_{s-}) dX_s^i, \quad t \geq 0.$$

**5.6. Corollary.** *Let  $A \in \mathcal{L}(H)$  be a bounded linear operator and let  $\sigma_1, \dots, \sigma_n : \mathbb{R}_+ \times H \rightarrow H$  be continuous. Assume there is constant  $L \geq 0$  such that (5.9) and (5.12) are satisfied. Then, for each  $h_0 \in H$ , there exists a unique strong adapted càdlàg solution  $(\Phi_t)_{t \geq 0}$  to (1.1) with  $\Phi_0 = h_0$  satisfying (5.13).*

*Proof.* The operator  $A$  is generated by the semigroup  $S_t = e^{tA}$ , which is pseudo-contractive, because

$$\|S_t\|_{\mathcal{L}(H)} \leq e^{t\|A\|_{\mathcal{L}(H)}}, \quad t \geq 0.$$

By Theorem 5.5, for each  $h_0 \in H$ , there exists a unique weak adapted càdlàg solution  $(\Phi_t)_{t \geq 0}$  to (1.1) with  $\Phi_0 = h_0$  satisfying (5.13), which also fulfills (5.15) by the boundedness of  $A$ , showing that  $(\Phi_t)$  is a strong solution to (1.1).  $\square$

## 6. STABILITY OF SOLUTIONS

Since one can never be sure about the accuracy of a proposed model, it is important to know how robust the model is concerning changes of the volatility and the initial condition. Therefore, we will be concerned with stochastic equations of the form

$$(6.1) \quad \begin{cases} d\Phi_t^m &= A\Phi_t^m dt + \sum_{i=1}^n \sigma_i^m(t, \Phi_{t-}^m) dX_t^i, \\ \Phi_0^m &= h_0^m \end{cases}$$

for  $m \in \mathbb{N}_0$ , and establish a stability result under appropriate regularity conditions.

As in the previous section, let  $H$  denote a separable Hilbert space,  $(S_t)_{t \geq 0}$  a  $C_0$ -semigroup in  $H$  with generator  $A$ , and let  $H_0 \subset H$  be a closed subspace such that  $(S_t)$  is pseudo-contractive on  $H_0$ . Let  $X = (X^1, \dots, X^n)$  be a vector of stochastic processes such that  $X^i \in \text{SPII}$  for  $i = 1, \dots, n$ .

In order to apply our existence and uniqueness result, Theorem 5.5, for the equations (6.1), we assume that  $\sigma_1^m, \dots, \sigma_n^m : \mathbb{R}_+ \times H \rightarrow H_0$  are continuous for each  $m \in \mathbb{N}_0$ , and that there is joint Lipschitz constant  $L \geq 0$  such that

$$(6.2) \quad \|\sigma_i^m(t, h_1) - \sigma_i^m(t, h_2)\| \leq L\|h_1 - h_2\|, \quad m \in \mathbb{N}_0$$

for all  $t \in \mathbb{R}_+$  and all  $h_1, h_2 \in H$ . Furthermore, we assume that  $X$  has no accumulating fixed times of discontinuity, and that (5.12) is valid. Then Theorem 5.5 ensures that for each  $m \in \mathbb{N}_0$  there exists a unique mild and a unique weak adapted càdlàg solution  $(\Phi_t^m)_{t \geq 0}$  to (6.1) with  $\Phi_0^m = h_0^m$  satisfying (5.13).

6.1. **Theorem.** *We assume that*

$$(6.3) \quad \mathbb{E}[\|h_0^m - h_0^0\|^2] \rightarrow 0,$$

$$(6.4) \quad \sup_{(t,h) \in \mathbb{R}_+ \times H} \|\sigma_i^m(t, h) - \sigma_i^0(t, h)\| \rightarrow 0, \quad i = 1, \dots, n$$

as  $m \rightarrow \infty$ . Then, for every  $T > 0$  we have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|\Phi_t^m - \Phi_t^0\|^2 \right] \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

*Proof.* Without loss of generality, we assume that  $T > 0$  is no fixed time of discontinuity for  $X$ . By (6.3) we obtain for  $m \rightarrow \infty$

$$(6.5) \quad \|S_t h_0^m - S_t h_0^0\|_{S^2[0, T]}^2 = \mathbb{E} \left[ \sup_{t \in [0, T]} \|S_t h_0^m - S_t h_0^0\|^2 \right] \leq e^{2\omega T} \mathbb{E} [\|h_0^m - h_0^0\|^2] \rightarrow 0.$$

Because of the uniform convergence (6.4) we get for all  $i = 1, \dots, n$

$$\sup_{t \in [0, T]} \|\sigma_i^m(t, \Phi_t^0) - \sigma_i^0(t, \Phi_t^0)\| \rightarrow 0 \quad \mathbb{P}\text{-a.s.}$$

The sequence  $\sup_{t \in [0, T]} \|\sigma_i^m(t, 0) - \sigma_i^0(t, 0)\|$  is, according to hypothesis (6.4), bounded by some constant  $C_T > 0$ . By (6.2), we deduce

$$\sup_{t \in [0, T]} \|\sigma_i^m(t, \Phi_t^0) - \sigma_i^0(t, \Phi_t^0)\|^2 \leq 2C_T^2 + 4L^2 \sup_{t \in [0, T]} \|\Phi_t^0\|^2.$$

The right-hand side of this inequality does not depend on  $m \in \mathbb{N}$ , and it has finite expectation, because  $\Phi^0 \in S^2[0, T]$ . Therefore, we deduce from Lebesgue's dominated convergence theorem for  $i = 1, \dots, n$

$$(6.6) \quad \|\sigma_i^m(t, \Phi_t^0) - \sigma_i^0(t, \Phi_t^0)\|_{S^2[0, T]} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Since  $X$  has no accumulating fixed times of discontinuity and by condition (5.12), there exists a decomposition  $0 = T_0 < T_1 < \dots < T_k = T$ , consisting only of no fixed times of discontinuity for  $X$ , such that (5.14) is valid.

We shall now prove by induction that  $\|\Phi^m - \Phi^0\|_{S^2[0, T_j]} \rightarrow 0$  for  $j = 0, \dots, k$ , which finishes the proof. For  $j = 0$  we have

$$\|\Phi^m - \Phi^0\|_{S^2[0, T_0]}^2 = \mathbb{E}[\|h_0^m - h_0^0\|^2] \rightarrow 0$$

for  $m \rightarrow \infty$  by assumption (6.3).

Now suppose that  $\|\Phi^m - \Phi^0\|_{S^2[0, T_j]} \rightarrow 0$  is valid. We obtain by the triangular inequality

$$\begin{aligned} \|\Phi^m - \Phi^0\|_{S^2[0, T_{j+1}]} &\leq \|S_t h_0^m - S_t h_0^0\|_{S^2[0, T_{j+1}]} \\ &+ \sum_{i=1}^n \left\| \int_0^t S_{t-s} (\sigma_i^m(s, \Phi_{s-}^m) - \sigma_i^0(s, \Phi_{s-}^0)) d(X^i)^{T_j} \right\|_{S^2[0, T_j]} \\ &+ \sum_{i=1}^n \left\| \int_0^t S_{t-s} (\sigma_i^m(s, \Phi_{s-}^m) - \sigma_i^0(s, \Phi_{s-}^0)) d(X^i - (X^i)^{T_j})_s \right\|_{S^2[0, T_{j+1}]} \end{aligned}$$

Since the variation function of  $X^i - (X^i)^{T_j}$  is given by (2.5), we arrive, by using (5.4) and (6.2), at

$$\begin{aligned} \|\Phi^m - \Phi^0\|_{S^2[0, T_{j+1}]} &\leq \|S_t h_0^m - S_t h_0^0\|_{S^2[0, T_{j+1}]} + L e^{\omega T_j} v_X(T_j) \|\Phi^m - \Phi^0\|_{S^2[0, T_j]} \\ &\quad + e^{\omega T_j} v_X(T_j) \sum_{i=1}^n \|\sigma_i^m(t, \Phi_t^0) - \sigma_i^0(t, \Phi_t^0)\|_{S^2[0, T_j]} \\ &\quad + L e^{\omega(T_{j+1}-T_j)} (v_X(T_{j+1}) - v_X(T_j)) \|\Phi^m - \Phi^0\|_{S^2[0, T_{j+1}]} \\ &\quad + e^{\omega(T_{j+1}-T_j)} (v_X(T_{j+1}) - v_X(T_j)) \sum_{i=1}^n \|\sigma_i^m(t, \Phi_t^0) - \sigma_i^0(t, \Phi_t^0)\|_{S^2[0, T_{j+1}]} \end{aligned}$$

Recalling (5.14), this inequality yields  $\|\Phi^m - \Phi^0\|_{S^2[0, T_{j+1}]} \rightarrow 0$  by relations (6.5), (6.6) and the induction hypothesis  $\|\Phi^m - \Phi^0\|_{S^2[0, T_j]} \rightarrow 0$ .  $\square$

## 7. CONCLUSION

We have established existence and uniqueness for infinite dimensional stochastic differential equations driven by processes with independent increments, see Theorem 5.5, with a focus on path regularity of the solution.

In contrast to other similar results in this area, the driving processes are allowed to be time-inhomogeneous and they may have fixed times of discontinuity. For processes with fixed times of discontinuity we have to accept the technical restriction (5.12), which is a joint condition on the Lipschitz constant and the distributions of the jump sizes of the driving process at fixed times of discontinuity.

Our method of proof slightly deviates from the techniques that are usually used in the literature. The intention is to obtain directly a solution with càdlàg trajectories, and therefore, we have worked on a smaller space of processes, such that each solution is automatically càdlàg. Since the Banach fixed point theorem only works on sufficiently small intervals, see Lemma 5.3, we have decomposed the interval  $[0, T]$  and inductively extended the solution, see Lemma 5.4, such that we have obtained a solution on the whole interval  $[0, T]$ .

Concerning the  $C_0$ -semigroup  $(S_t)$ , we have assumed that it is pseudo-contractive, at least on a closed subspace. This is a standard assumption, which is required, if one is interested in path properties of the solution, cf. [1, 8]. Indeed, the pseudo-contractivity of  $(S_t)$  is required for an application of the Kotelenetz theorem, and it also needed for the Szekőfalvi-Nagy's theorem on unitary dilations. If one is merely interested in the existence of a solution, without focus on path properties, no conditions on the  $C_0$ -semigroup have to be imposed, see e.g. [10, 17].

Finally, we have proven stability of the solutions for such equations, see Theorem 6.1. We remark that, due our choice of the norm on the space  $S^2[0, T]$ , we cannot apply usual tools like Gronwall's lemma. Instead, our method of decomposing the interval  $[0, T]$  into sufficiently small subintervals, has worked out here as well.

## REFERENCES

- [1] Albeverio, S., Mandrekar, V., Rüdiger, B. (2006): Existence of mild solutions for stochastic differential equations and semilinear equations with non Gaussian Lévy noise. Preprint no. 314, SFB 611, University of Bonn.
- [2] Da Prato, G., Zabczyk, J. (1992): *Stochastic equations in infinite dimensions*. New York: Cambridge University Press.
- [3] Davies, E. B. (1976): *Quantum theory of open systems*. London: Academic Press.
- [4] Eberlein, E., Jacod, J., Raible, S. (2005): Lévy term structure models: no-arbitrage and completeness. *Finance and Stochastics* **9**, 67–88.
- [5] Eberlein, E., Kluge, W. (2006): Exact pricing formulae for caps and swaptions in a Lévy term structure model. *Journal of Computational Finance* **9**(2), 99–125.
- [6] Eberlein, E., Kluge, W. (2006): Valuation of floating range notes in Lévy term structure models. *Mathematical Finance* **16**, 237–254.
- [7] Engel, K.-J., Nagel, R. (2000): *One-parameter semigroups for linear evolution equations*. New York: Springer.
- [8] Filipović, D., Tappe, S. (2007): Existence of Lévy term structure models. Forthcoming in *Finance and Stochastics*.
- [9] van Gaans, O. (2005): A series approach to stochastic differential equations with infinite dimensional noise. *Integral Equations and Operator Theory* **51**(3), 435–458.
- [10] van Gaans, O. (2005): Invariant measures for stochastic evolution equations with Lévy noise. Technical Report, Leiden University.
- [11] Hausenblas, E. (2005): SPDEs driven by Poisson random measure: Existence and uniqueness. *Electronic Journal of Probability* **11**, 1496–1546.
- [12] Hausenblas, E. (2007): SPDEs driven by Poisson random measure with non Lipschitz coefficients: Existence results. Forthcoming in *Probability Theory and Related Fields*.
- [13] Hausenblas, E., Seidler, J. (2001): A note on maximal inequality for stochastic convolutions. *Czechoslovak Mathematical Journal* **51**(126), 785–790.
- [14] Hausenblas, E., Seidler, J. (2007): Stochastic convolutions driven by martingales: Maximal inequalities and exponential integrability. forthcoming in *Stoch. Anal. Appl.*
- [15] Jacod, J., Shiryaev, A. N. (1987): *Limit theorems for stochastic processes*. Berlin: Springer.
- [16] Knoche, C. (2004): SPDEs in infinite dimensions with Poisson noise. *Comptes Rendus Mathématique. Académie des Sciences. Paris, Serie I* **339**, 647–652.
- [17] Knoche, C. (2005): Mild solutions of SPDEs driven by Poisson noise in infinite dimensions and their dependence on initial conditions. PhD thesis, University of Bielefeld.
- [18] Kotelenez, P. (1982): A submartingale type inequality with applications to stochastic evolution equations. *Stochastics* **8**, 139–151.
- [19] Marinelli, C. (2007): Local well-posedness of Musiela’s SPDE with Lévy noise. Preprint no. 329, SFB 611, University of Bonn.
- [20] Métivier, M. (1982): *Semimartingales*. Walter de Gruyter, Berlin.
- [21] Peszat, S., Zabczyk, J. (2007): *Stochastic partial differential equations with Lévy noise: Evolution equations approach*. Cambridge University Press, Cambridge. *to appear*.
- [22] Peszat, S., Zabczyk, J. (2007): Heath-Jarrow-Morton-Musiela equation of bond market. Preprint IMPAN 677, Warsaw.
- [23] Protter, P. (2005): *Stochastic integration and differential equations*. Second Edition, Version 2.1, Berlin: Springer.
- [24] Rüdiger, B. (2004): Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces. *Stoch. Stoch. Rep.* **76**(3), 213–242.
- [25] Szekőfalvi-Nagy, B., Foiaş, C. (1970): *Harmonic analysis of operators on Hilbert space*. North-Holland, Amsterdam.