Set-Valued Risk Measures and Systemic Risk

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May 28, 2014
1. Overview

2. Financial networks

3. Systemic risk measures
   - Eisenberg-Noe network model

4. Computation

5. Orthant risk measures
Idea: Capital requirements for financial firms to control the risk to the outside economy
1. Overview

- Idea: Capital requirements for financial firms to control the risk to the outside economy
- Model the financial system via a network of obligations
- Introduce (random) stresses into the system and find payment structure
- System is “acceptable” as measure of net payments to the outside economy (1 dimensional), but capital requirements separated by institution
2. Financial networks

Figure: Independent financial firms
Figure: Network with single systemically important firm
2. Financial networks

Figure: Network with no clear systemically important firm
2. Financial networks

Figure: Network with node for “real” economy
3. Systemic risk measures

- $n$ financial firms
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- Equity and loss (E&L) function: $e : \mathbb{R}_+^n \rightarrow \mathbb{R}^{n+1} \cup \{-\infty\}$
- Pre-image: Vector of bank endowments before network effects
- Image: Vector of bank equity after network effects
3. Systemic risk measures

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- $e_0$ is the equity value of the outside economy from the financial system
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- Pre-image: Vector of bank endowments before network effects
- Image: Vector of bank equity after network effects
- \( e_0 \) is the equity value of the outside economy from the financial system
- Assume:
  - \( e \) is nondecreasing
  - \( e(y) = -\infty \) for all \( y \notin \mathbb{R}^n_+ \)
  - \( e_0 \) is bounded from above
  - \( e_0(y) \geq 0 \) for all \( y \in \mathbb{R}^n_+ \)
3. Systemic risk measures

Systemic Risk Measures

\[ R^{sys}_A : L^0(\mathbb{R}^n) \rightarrow \mathcal{P}(\mathbb{R}^n; \mathbb{R}_+^n) = \{ D \subseteq \mathbb{R}^n \mid D = D + \mathbb{R}_+^n \} \] is a **systemic risk measure** if for some acceptance set \( A \subseteq L^\infty(\mathbb{R}) \) of a scalar risk measure:

\[ R^{sys}_A(X) = \{ k \in \mathbb{R}^n \mid e_0(k + X) \in A \}. \]
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Equivalently:

\[ R_{A}^{sys}(X) = \{ k \in \mathbb{R}^{n} \mid k + X \in \mathcal{A}^{e} \} \]

\[ \mathcal{A}^{e} = e_{0}^{-1}[A] := \{ Y \in L^{0}(\mathbb{R}^{n}_{+}) \mid e_{0}(Y) \in A \} \]
Assume: \( e_0 \) is concave and continuous
\( \mathcal{A} \) is convex, closed, and law-invariant
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$\mathcal{A}$ is convex, closed, and law-invariant

Properties: Let $X, Y \in L^0(\mathbb{R}^n)$, $k \in \mathbb{R}^n$, and $\alpha \in [0, 1]$

- **Translative:** $R^\text{sys}_\mathcal{A}(X + k) = R^\text{sys}_\mathcal{A}(X) - k$
- **Monotone:** $R^\text{sys}_\mathcal{A}(X) \supseteq R^\text{sys}_\mathcal{A}(Y)$ if $X \geq Y$ a.s.
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- **Law-invariant**: $R^\text{sys}_A(X) = R^\text{sys}_A(Y)$ if $X \overset{d}{=} Y$
3.1 Systemic risk measures: Eisenberg-Noe network model

\[ \bar{p}_{1,2}, \bar{p}_{1,3}, \bar{p}_{3,1}, \bar{p}_{4,5}, \bar{p}_{4,6}, \bar{p}_{5,1}, \bar{p}_{5,4}, \bar{p}_{6,3}, \bar{p}_{6,4}, \bar{p}_{3,2} \]
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- Firm $i$ has endowment $x_i$
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- Liability of firm $i$ to outside economy is given by $b_i \geq 0$
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- Liability of firm $i$ to $j$ is given by $\bar{p}_{ij} \geq 0$
- Liability of firm $i$ to outside economy is given by $b_i \geq 0$
- Total liabilities for firm $i$ given by $\bar{p}_i = b_i + \sum_{j \neq i} \bar{p}_{ij}$
- Relative liabilities for firm $i$ to $j$ is given by $a_{ij} = \frac{\bar{p}_{ij}}{\bar{p}_i}$
Realized clearing payment given endowments $x$ is provided by the fixed point problem:

$$p_i(x) = \bar{p}_i \land \left( \sum_{j=1}^{n} a_{ji} \cdot p_j(x) + x_i \right)$$
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- Realized sum of debt and equity minus promised payments

$$e_i(x) := x_i + \sum_{j=1}^{n} a_{ji} \cdot p_j(x) - \bar{p}_i$$

is the value of firm $i$ (if positive) or losses from default (if negative)
3.1 Systemic risk measures: Eisenberg-Noe network model

- Net payment to outside economy given by:

\[
e_0(x) := \sum_i \frac{b_i}{\bar{p}_i} \cdot p_i(x)
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- E&L function \( e \) is concave, nondecreasing, and Lipschitz continuous
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- \( e_0 \) is concave, nondecreasing, and Lipschitz continuous
4. Computation

- To compute: approximate expectations by Monte Carlo simulation
- Approximate via smart grid search for boundary of set
- Idea: draw a grid over area of interest (e.g. box around $C(X)$), and find the grid points in the set
- Possible improvement with parallel computing
Sample Acceptance Sets:

- *Average value at risk*: for $\lambda \in (0, 1)$

$$
\mathcal{A}^\lambda = \{ Z \in L^\infty(\mathbb{R}) \mid \inf_{r \in \mathbb{R}} (\mathbb{E} [(r - Z)^+] - r \lambda) \leq 0 \}
$$
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- **Average value at risk:** for $\lambda \in (0, 1)$

$$\mathcal{A}^\lambda = \{ Z \in L^\infty(\mathbb{R}) \mid \inf_{r \in \mathbb{R}} (\mathbb{E}[ (r - Z)^+] - r \lambda) \leq 0 \}$$

- **Utility-based shortfall risk:** for convex loss function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ and threshold $z \in \mathbb{R}$

$$\mathcal{A}^{\ell,z} = \{ Z \in L^\infty(\mathbb{R}) \mid \mathbb{E}[\ell(-Z)] \leq z \}$$
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  \]

- **Optimized certainty equivalents**: for concave utility $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

  \[
  \mathcal{A}^u = \{ Z \in L^\infty(\mathbb{R}) \mid \sup_{\eta \in \mathbb{R}} (\eta + \mathbb{E} [u(Z - \eta)]) \geq 0 \}
  \]
Figure: Grid search
4. Computation

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$k = (0, 0)$
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$e_0(X + k) \in \mathcal{A}$?

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Figure: Centrally connected network

Figure: Completely connected network
4. Computation

Figure: Centrally connected (×2) vs. completely connected
Figure: Centrally connected vs. completely connected
Figure: Centrally connected vs. completely connected
Figure: Centrally connected: independent vs. comonotonic
5. Orthant risk measures

- Systemic risk measure $R^{sys}_A(X)$ provides the set of all capital allocations $k$ making $k + X$ acceptable.
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- Systemic risk measure \( R_{\mathcal{A}}^{sys}(X) \) provides the set of all capital allocations \( k \) making \( k + X \) acceptable.
- In practice, want allocation \( k^* \) to be a minimizer of the set

\[
k^* \in \text{Min } R_{\mathcal{A}}^{sys}(X) := \{ k \in \mathbb{R}^n : (k - \mathbb{R}_+^n) \cap R_{\mathcal{A}}^{sys}(X) = \{ k \} \}
\]

- Capital requirements for each firm \( k_i \geq k_i^* \).
5. Orthant risk measures

- Systemic risk measure $R_{\mathcal{A}}^{sys}(X)$ provides the set of all capital allocations $k$ making $k + X$ acceptable.
- In practice, want allocation $k^*$ to be a minimizer of the set
  \[ k^* \in \text{Min } R_{\mathcal{A}}^{sys}(X) := \{ k \in \mathbb{R}^n : (k - \mathbb{R}_+) \cap R_{\mathcal{A}}^{sys}(X) = \{k\} \} \]
- Capital requirements for each firm $k_i \geq k_i^*$, i.e. capital requirements are $k^* + \mathbb{R}_+$.
Orthant risk measure

$k_{sys}^{A} : L^0(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is an orthant risk measure associated with the systemic risk measure $R_{sys}^A$ if for all $X, Y \in L^0(\mathbb{R}^n)$, $k \in \mathbb{R}^n$, and $\alpha \in [0, 1]$:

- **Minimal valued:** $k_{sys}^{A} (X) \in \text{Min} R_{sys}^A (X)$
- **Translative:** $k_{sys}^{A} (X + k) = k_{sys}^{A} (X) - k$
- **Monotone:** $k_{sys}^{A} (X) + \mathbb{R}^n_+ \ni k_{sys}^{A} (Y)$ if $X \geq Y$ a.s.
- **Quasi-convex:**
  
  \[ k_{sys}^{A} (\alpha X + (1 - \alpha)Y) + \mathbb{R}^n_+ \ni k_{sys}^{A} (X) \lor k_{sys}^{A} (Y) \]

- **Law-invariant:** $k_{sys}^{A} (X) = k_{sys}^{A} (Y)$ if $X \overset{d}{=} Y$
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- Fix $w \in \text{int}(\bigcap_Z \text{recc} \left( R_A^{sys}(Z) \right)^+) \ (\text{typically } w \in \mathbb{R}^{n}_{++})$

$$
k^{sys}_A(X) = \arg \min \left\{ \sum_{i=1}^{n} w_i \cdot k_i \mid k \in R_A^{sys}(X) \right\}
$$

defines a orthant risk measure

- Options:
  - $w = (1, ..., 1)$: minimize system-wide addition of capital
5. Orthant risk measures

- Fix $w \in \text{int}(\bigcap_Z \text{recc} \left( R^s_{\mathcal{A}}(Z) \right)^{+})$ (typically $w \in \mathbb{R}^{n}_{++}$)

$$k^s_{\mathcal{A}}(X) = \arg \min \left\{ \sum_{i=1}^{n} w_i \cdot k_i \mid k \in R^s_{\mathcal{A}}(X) \right\}$$

defines an orthant risk measure

- Options:
  - $w = (1, \ldots, 1)$: minimize system-wide addition of capital
  - $w_i = 1/\bar{p}_i$: minimize total capital weighted by obligation
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- Fix \( w \in \text{int}(\bigcap \mathbb{Z} \text{recc} (R^s_{\mathcal{A}}(Z))^+) \) (typically \( w \in \mathbb{R}^n_+ \))

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\kappa^s_{\mathcal{A}}(X) = \arg \min \left\{ \sum_{i=1}^{n} w_i \cdot k_i \mid k \in R^s_{\mathcal{A}}(X) \right\}
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defines a orthant risk measure

- Options:
  - \( w = (1, ..., 1) \): minimize system-wide addition of capital
  - \( w_i = 1/\bar{p}_i \): minimize total capital weighted by obligation
  - \( w_i = \max(1/AV@R_{0.5\%}(X_i + \sum_{j \neq i} \bar{p}_j \cdot a_{ji} - \bar{p}_i), \epsilon) \): minimize total capital weighted by individual risk (neglecting counterparty risk)
Thank you