On Asymptotic Power Utility-Based Pricing and Hedging

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Outline

Introduction

Pricing and Hedging

Given:

- Risk-free bond S^0 normalized to 1
- Discounted stock price process modeled by semimartingale S
- H: Random payoff, e.g., **option** written on S

Classical problems of Mathematical Finance:

- Reasonable **price** for H?
- How to **hedge** the resulting risk by dynamic trading in S^0, S?

Introduction

Pricing and hedging in incomplete markets?

Complete markets:

- Any payoff is replicable $\Rightarrow$ perfect hedging strategy.
- Unique price compatible with No Arbitrage.

Incomplete markets:

- Incompleteness caused by, e.g., jumps or stochastic volatility.
- Replication no longer possible.
- Many different prices consistent with No Arbitrage.

Additional criterion for pricing and hedging?

Introduction

Martingale modeling

Popular approach in practice:

- Model liquid primary securities directly under EMM Q.
- Existence guaranteed by FTAP.
- Price illiquid claims by their Q-expectation.
- Yields consistent, arbitrage-free prices.
- But:
  - Unique only in complete markets.
  - Extrapolates to non-traded claims.
  - Ignores residual risk.
  - Says nothing about hedging.
- How to price hedging errors in incomplete markets?

Introduction

Mean-variance hedging

Popular approach in Mathematical Finance:

- Replication impossible $\Rightarrow$ minimize expected squared hedging error:

$$(v, \varphi) \mapsto E((\underbrace{v + \varphi \bullet S_T}_{:= V_T(\varphi)} - H)^2)$$

- Hedge: minimizer φ
- Price: minimizer v plus some(?) function of hedging error.
- Advantage: analytically tractable.
- Disadvantage: economically questionable. Gains and losses punished alike.

Economically better founded alternative?

Use *increasing* utility function, maximize expected utility:

- Without options:

$$U(v) := \sup_{\varphi} E(u(v + \varphi \bullet S_T)) \qquad (\star)$$

- After selling q options H for π^q each:

$$U^q(v + q\pi^q) := \sup_{\varphi} E(u(v + q\pi^q + \varphi \bullet S_T - qH)) \qquad (\star\star)$$

Indifference price: threshold π^q for which $U(v) = U^q(v + q\pi^q)$.

Utility-based hedge: difference between optimizers φ^q in $(\star\star)$ and $\widehat{\varphi}$ in $(\star)$.

- Advantage: economically plausible.
- Disadvantage: computation usually impossible
- Way out: **first-order approximations** for small number of claims ($q \to 0$):

$$\pi^q = \pi^0 + q\pi' + o(q^2)$$
$$\varphi^q = \widehat{\varphi} + q\varphi' + o(q^2)$$

- $\widehat{\varphi}$: optimal strategy for pure investment problem
- π^0: expectation under dual EMM $dQ_0/dP \sim u'(V_T(\widehat{\varphi}))$ [Davis (1997), Karatzas and Kou (1996)]

$\Rightarrow$ What about **hedge** φ' and **risk premium** π'?

Asymptotic Utility-Based Pricing and Hedging

The results of Kramkov and Sîrbu

Goal: first-order approximations

$$\pi^q = \pi^0 + q\pi' + o(q^2), \quad \varphi^q = \widehat{\varphi} + q\varphi' + o(q^2)$$

Kramkov & Sîrbu (2006,2007) for utilities on $\mathbb{R}_+$, Sirbû (2010) on $\mathbb{R}$: if **risk-tolerance wealth process** R exists with

$$R_T = -\frac{u'(V_T(\widehat{\varphi}))}{u''(V_T(\widehat{\varphi}))},$$

then:

- φ': mean-variance optimal hedge
- π': multiple of corresponding expected squared hedging error
- **But**: relative to numeraire R and under adjusted dual EMM Q_0, i.e. under $dQ^{\$}/dQ_0 \sim V_T(\widehat{\varphi})$

Asymptotic Utility-Based Pricing and Hedging

The Results of Kramkov and Sîrbu ct'd

Asymptotic utility-based hedging:

- Mean-variance hedging strategy.
- Limiting price is expectation under specific EMM.
- Risk premium for incompleteness is squared hedging error.
- But: computed under marginal pricing measure, and relative to numeraire given by the optimal wealth process for the pure investment problem.
- Interpretation: any utility function is locally quadratic around the optimum.
- Tractable examples?

CARA, i.e., Exponential utility $u(x) = -\exp(-px)$:

- Constant risk-tolerance wealth process replicating

$$R_T = -u'(V_T(\widehat{\varphi}))/u''(V_T(\widehat{\varphi})) = p$$

- Hence: mean-variance hedging under Minimal Entropy Martingale Measure, w.r.t. original numeraire.
- Compare Mania & Schweizer (2005), Becherer (2006), and Kallsen & Rheinländer (2009) for continuous asset prices.
- As tractable as mean-variance hedging for Lévy and some affine models [Kallsen, Rheinländer & Vierthauer (2010)].

What about CRRA, i.e., power utility $u(x) = x^{1-p}/(1-p)$?

For **CRRA**, i.e., power utility $u(x) = x^{1-p}/(1-p)$:

- Risk tolerance replicated by scaled optimal wealth process:

$$R_T = -u'(V_T(\widehat{\varphi}))/u''(V_T(\widehat{\varphi})) = pV_T(\widehat{\varphi})$$

- Hence: mean-variance hedging under q-optimal martingale measure. Additional change of numeraire.
- As for mean-variance hedging à là Gourieroux et al. (1998).
- In principle feasible for Lévy and some affine models.
- But: additional redundant asset:

$$(1, S^{\$}) := \left(1, \frac{1}{V(\widehat{\varphi})/v}, \frac{S}{V(\widehat{\varphi})/v}\right) \quad \text{instead of} \quad (1, S)$$

- Does not allow to apply results from the mean-variance literature directly. Complicates interpretation.

Asymptotic Utility-Based Pricing and Hedging

An alternative representation

- Kramkov & Sîrbu (2007): Hedge φ' minimizes

$$E_{Q^\$}\left(\left(\pi^{0\$}+\psi'\bullet S_T^\$-H^\$\right)^2\right)=E_{Q^\$}\left(\left(\frac{\pi^0+\psi'\bullet S-H}{V(\widehat{\varphi})/v}\right)^2\right)$$

  over all strategies ψ'.

- **Idea**: Equivalent to minimizing

$$E_{P^€}\left(\left(\pi^0+\psi'\bullet S_T-H\right)^2\right)\quad \text{for} \quad \frac{dP^€}{dQ^\$}=\frac{1}{(V(\widehat{\varphi})/v)^2}$$

  $\Rightarrow$ Mean-variance hedging under auxiliary measure $P^€$ w.r.t original numeraire!

Asymptotic Utility-Based Pricing and Hedging

An alternative representation

Disadvantage of alternative approach:

- $P^{€}$ typically is not an EMM $\Rightarrow$ harder hedging problem.

Advantages of alternative approach:

- Original numeraire.
- Černý & Kallsen (2007): solution via Föllmer-Schweizer decomposition after suitable change of measure.
- New measure already determined by solution to pure investment problem.
- Hence: same complexity as for mean-variance hedging in the martingale case.
- Results from the literature directly applicable.

But: Delicate technical obstacle!

Asymptotic utility-based pricing and hedging

An alternative approach

Reconsider

$$\min\left\{E_{Q^{\$}}\left(\left(\pi^{0\$}+\psi'\bullet S^{\$}_T-H^{\$}\right)^2\right):\psi'\text{ admissible}\right\}$$

$$\overset{?}{\Leftrightarrow}\min\left\{E_{P^{€}}\left(\left(\pi^{0}+\psi'\bullet S_T-H\right)^2\right):\psi'\text{admissible}\right\}$$

Technical problem:

- Admissibility not invariant under change of numeraire.
- Only equivalent, if the process $\widehat{\varphi}\bullet S$ that links $Q^{\$}$ and $P^{€}$ is a martingale under **any** EMM.
- Typically impossible to check even in concrete models.
- No reason why this should hold in general.

So how to make the heuristic argument precise?

Asymptotic utility-based pricing and hedging

An alternative approach ct'd

Characterization of mean-variance hedging problem by Černý & Kallsen (2007) consists of two parts:

- Local characterization of candidates via semimartingale characteristics.
- Global admissibility conditions that ensure optimality.

Key idea:

- Admissibility not satisfied, but also not needed.
- First-order terms from Kramkov and Sîrbu (2006, 2007) characterized by local conditions of Černý & Kallsen (2007).
- Interpretation as mean-variance hedging problem requires extra assumptions, but is not needed to apply formulas.
- Key tool for derivation: semimartingale calculus. Does not require global assumptions.

Asymptotic Utility-Based Pricing and Hedging

An alternative approach ct'd

In summary: for power utility-based pricing and hedging...

- Start from optimal wealth process $V(\widehat{\varphi})$ for pure investment problem.
- Limiting price for small claims is expectation under dual EMM Q_0 with density $\sim V_T(\widehat{\varphi})^{-p}$.
- First-order correction is minimal squared hedging error under measure $P^{€}$ with density $\sim V_T(\widehat{\varphi})^{-1-p}$.
- Asymptotic hedging strategy is corresponding mean-variance hedge.
- Tractable examples?
  - Need tractable pure investment problem.
  - Need "nice" structure under Q_0 and $P^{€}$.
  - OK for some "affine" models.

Application to affine models

Affine stochastic volatility models

Activity v and log-price X modeled as bivariate **affine process**:

$$E\left(e^{iu_1 v_T + iu_2 X_T}\middle|\mathscr{F}_t\right) = e^{\Psi_0(t,T,iu)+\Psi_1(t,T,iu)v_t+\Psi_2(t,T,iu)X_t}$$

- Thoroughly analyzed by Duffie et al. (2003).
- Flexible and tractable
- Example: OU-time change model of Carr et al. (2003):

$$\begin{aligned} dv_t &= -\lambda v_t dt + dZ_t \\ X_t &= L_{\int_0^t v_s ds} \end{aligned}$$

for Lévy process L, subordinator Z.

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Application to Affine Models

Asymptotic utility-based pricing and hedging

Step 1: Solve the pure investment problem.

- Computation though appropriate ansatz.
- Verification via Martingale Optimality Principle.

Step 2: Mean-variance hedging under $P^{€}$.

- Need: tractable model (e.g., Lévy, affine) under $P^{€}$.
- Works for Lévy and some affine models under P.
  - Wealth process $V(\widehat{\varphi})$ needs to be exponentially affine.
  - Requires excess return proportional to local variance. Satisfied for time-change models.
  - Then: density processes given by moments. Again affine by transform formula. Change of measure retains affine structure.
- In this case: first-order approximations given by formulas from Hubalek et al. (2006) resp. Kallsen & Vierthauer (2009).

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Application to Affine Models

Example: utility-based hedges in OU time-change model

Numerical example:

- Returns follow NIG Lévy process in business time.
- Time change to calendar time given by Gamma-OU process.
- Parameters estimated from 20 years of DAX data.
- Skewness: -0.4. Excess kurtosis: 5.8.
- Evaluation of the integral-transform formulas from Kallsen & Vierthauer (2009) by numerical quadrature.
- European call option with payoff $H = (S_{0.25} - 100)^{+}$.

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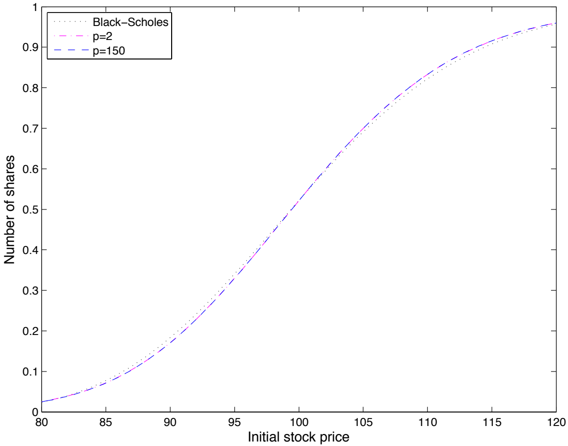
Application to Affine Models

Example: utility-based hedges in OU time-change model

Hedges for varying initial stock prices, risk aversion:



Application to Affine Models

Example: Utility-Based Hedges in OU time-change model ct'd

Asymptotic power utility-based hedges:

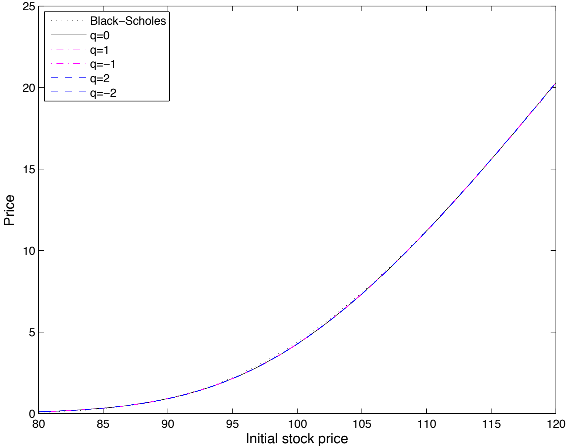
- Almost independent of risk aversion.
- Very close to both Black-Scholes and exponential hedge (limit for high risk aversion, $p \to \infty$).
- Incompleteness, preferences do not cause big deviation from Black-Scholes.
- Delta-hedging is surprisingly robust even with jumps and stochastic volatility [compare Denkl et al. (2012)].

What about price corrections?

Application to affine models

Example: Utility-based prices in OU time-change model ct'd

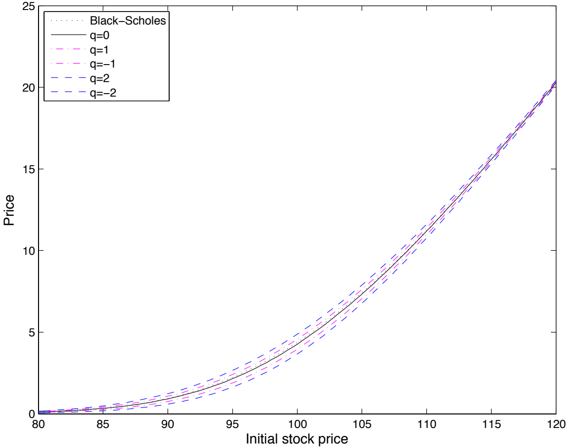
For low risk aversion $p = 2$:



Application to Affine Models

Example: utility-based prices in OU time-change model ct'd

For high risk aversion $p = 150$:



Application to Affine Models

Example: utility-based prices in OU time-change model ct'd

Asymptotic power utility-based prices:

- Very close to Black-Scholes for risk aversions as in most of the economic literature.
- In particular, bid- and ask prices typically on the same side.
- For much larger risk aversions: bid-ask spread above and below Black-Scholes price.
- With estimated parameters, model incompleteness due to jumps and stochastic volatility can explain large option spreads only with very high risk aversion.

Summary

Asymptotic utility-based pricing and hedging

To compute first-order approximations

$$\pi^q = \pi^0 + q\pi' + o(q^2), \quad \varphi^q = \widehat{\varphi} + q\varphi' + o(q^2)$$

1. Solve the pure investment problem $\max_\psi E(u(V_T(\psi)))$.
2. Apply local characterizations for the mean-variance hedging problem of the claim under $dP^{€}/dP \sim V_T(\widehat{\varphi})^{-1-p}$.

- Step 1 is a classical problem, more or less explicit solutions in a wide range of Markovian models.
- Step 2 is easier than mean-variance hedging under $P^{€}$, since one does not have to verify admissibility of $\widehat{\varphi}$.
- Semi-explicit, numerically tractable formulas for Lévy and some affine models.