A Continuous Time Approximation of an Evolutionary Stock Market Model

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Abstract

We derive a continuous time approximation of the evolutionary market selection model of Blume & Easley (1992). Conditions on the payoff structure of the assets are identified that guarantee convergence. We show that the continuous time approximation equals the solution of an integral equation in a random environment. For constant asset returns, the integral equation reduces to an autonomous ordinary differential equation. We analyze its long-run asymptotic behavior using techniques related to Lyapunov functions, and compare our results to the benchmark of profit-maximizing investors.

Key words: Portfolio theory, evolutionary finance, continuous time Euler approximation, stochastic processes in random environments, Lyapunov function

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1 Introduction

The axiom of profit maximization is a cornerstone of neoclassical economics. Often it is justified by the market selection hypothesis, which argues that maximization describes the long-run market behavior induced by an evolutionary selection process, cf. Friedman (1953) and Fama (1965). While intuitively appealing, this argument clearly needs a rigorous analysis.

An explicit model for the market selection mechanism has been proposed in a seminal paper by Blume & Easley (1992). In an asset model with endogenous prices in discrete time, agents follow simple trading strategies. They keep the proportion of wealth invested in each asset fixed over time and reinvest their payoffs. The market process induces a redistribution of wealth among traders. Blume & Easley (1992) investigate the long-run dynamics of the selection process. Under strong conditions on the underlying random variables and the payoff structure of the assets they identify the unique survivor of the market selection process.

This result has recently been generalized. Evstigneev, Hens & Schenk-Hoppé (2002) extend the model to a more complex payoff structure for the case that uncertainty is modeled by a sequence of independent

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random variables taking values in a finite state space. They identify the unique surviving strategy. For general ergodic states, Hens & Schenk-Hoppé (2004) derive local stability results.

In the current paper, we provide a continuous time approximation for the model of Blume & Easley (1992) for general random payoffs of the assets. Here, we assume that trading takes place at a higher frequency and that in each trading interval agents reinvest only a fraction of their wealth. If the payoffs of the assets converge nicely (as the time between two successive trading dates approaches zero), then also the wealth process of the agents converges by a functional limit theorem which is closely related to the well known Euler scheme. The continuous time limit of the wealth process equals the solution of a non linear integral equation in a random environment.

The continuous time approximation of the wealth process relies on a proper convergence of the payoffs of the assets, as the length of the trading intervals tends to zero. We suggest an economically meaningful model for the dividend processes and their convergence. Dividend payments are modeled as increments of stochastic firm value processes. Conditions on these processes are identified, which ensure the applicability of the functional limit theorem. For this purpose, the notion of locally finite kernels turns out to be useful.

In a further step, we analyze the long-run asymptotic behavior of the continuous time approximation in the simplest special case. Namely, we assume that the dividend process of the assets is deterministic and constant. The Markovian case will be investigated in Buchmann & Weber (2005). For constant dividend payments, the deterministic dynamics of the wealth process in continuous time is described by a non linear, autonomous ordinary differential equation. We characterize its long-run asymptotic behavior. Here, we employ the technique of Lyapunov functions. In particular, we prove that there exists a unique strategy that asymptotically gathers all wealth in any market without redundant trading strategies. The strategy consists in dividing income proportionally to relative payoffs of the assets.

Finally, we compare these results to a Walrasian equilibrium of myopic agents who are price takers. In continuous time, the investors' objectives coincides with the growth optimality of their strategies. The equilibrium solutions are closely connected to the asymptotic behavior of the evolutionary model.

Evolutionary models of portfolio selection are related to the literature on growth optimal portfolios, see e.g. Hakansson (1970), Thorp (1971), Algoet & Cover (1988), Cover (1991), Hakansson & Ziemba (1995), Browne & Whitt (1996), Karatzas & Shreve (1998), and Aurell, Baviera, Hammarlid, Serva & Vulpiani (2000). As common in mathematical finance and in contrast to the evolutionary approach, these models usually assume an exogenous price process. Equilibrium consequences are neglected in these models. The current model makes a connection between an evolutionary approach and continuous time processes which are commonly used in mathematical finance. This has two implication. Techniques from stochastic analysis can be used for the investigation of the proposed model. At the same time, equilibrium effects are treated endogenously.

The balance of this paper is organized as follows. In Section 2 we present the discrete time model of dynamic asset allocation of Blume & Easley (1992). In Section 3 we provide a continuous time approximation of the wealth process and suggest an economically meaningful model for the dividend processes. In Section 4 we study the long-run asymptotic behavior of the continuous time approximation of the wealth process in the deterministic case and examine a rational benchmark. Section 5 concludes.

2 Modeling Dynamic Asset Allocation

2.1 The economy

In this section we provide a model of dynamic portfolio allocation and the evolution of wealth of investors in a financial market. By $i \in I = \{1, 2, \ldots, I\}$ we denote a finite set of investors who can invest into assets $k \in K = \{1, 2, \ldots, K\}$ at discrete points in time $t \in \mathbb{N}$.

At time t, investor $i \in I$ is endowed with wealth $w_i^t \in \mathbb{R}_+$. For the vector of agents' wealth we will write $w^t = (w_i^t)_{i \in I}$. At each point in time t each investor i acquires a portfolio $a_i^t = (a_{i,1}^t, a_{i,2}^t, \ldots, a_{i,K}^t)$; here $a_{i,k}^t$ denotes the number of shares of asset k in the portfolio. For simplicity, we assume that assets live only for one period and are re-born at every period. Denoting the price of one share of asset k by ρ_k^t, the I budget constraints of the investors $i \in I$ can be written in the following form:

$$w_i^t = \sum_{k=1}^{K} \rho_k^t \cdot a_{i,k}^t \tag{1}$$

The prices are determined in a Walrasian market by the K equilibrium equations

$$\bar{a}_k^t = \sum_{i=1}^{I} a_{i,k}^t \tag{2}$$

where $\bar{a}_k^t > 0$ is the total supply of asset k in period t. For simplicity, we suppose that the supply of each asset does not depend on time and is non-random. By an appropriate renormalization of the payoffs of the assets we may and will assume that $\bar{a}_k^t \equiv 1$ for all $k \in K$. Economically, this hypothesis could be expressed in terms of a stock split. The *budget shares* of the assets in the portfolio of the investors are given by

$$\lambda_{i,k}^t = \frac{\rho_k^t \cdot a_{i,k}^t}{w_i^t} \tag{3}$$

The sequence of budget shares $\lambda_i = (\lambda_i^t)_{t \in \mathbb{N}} = (\lambda_{i,1}^t, \lambda_{i,2}^t, \ldots, \lambda_{i,K}^t)_{t \in \mathbb{N}}$ will be called the *trading strategy* of investor i.

Rewriting (2), we obtain the following equation for the market-clearing price:

$$\rho_k^t = \sum_{i=1}^{I} \lambda_{i,k}^t \cdot w_i^t \tag{4}$$

The shares bought at time t pay a dividend at time $t+1$ which we will assume to be random. We let $(\Omega, \mathcal{F}, P)$ be a probability space. By $A_k^{t+1} : \Omega \to \mathbb{R}_+$ we denote the dividend payment of asset k at time $t+1$. We will assume that all random quantities under consideration, i.e. A_k^t, $a_{i,k}^t$, and w_i^t ($i \in I$, $k \in K$, $t \in \mathbb{N}$), are measurable.

Total dividend payments received by agents i at time $t+1$ can be calculated as

$$D_i^{t+1} = \sum_k a_{i,k}^t \cdot A_k^{t+1} \tag{5}$$

The quantities we considered so far were given by their nominal value. The real wealth of any investor must be described as a fraction of total wealth times the real value of the economy. To keep the analysis simple, we will abstract from growth and assume that the real value of the economy is constant over time and equal to 1. Hence, in real terms economic quantities are given by choosing total market wealth as numeraire. Real wealth of investor i at time t is thus given by relative wealth

$$r_i^t = \frac{w_i^t}{\sum_{j=1}^I w_j^t} \tag{6}$$

Normalizing the prices of the assets by the market wealth we obtain the real prices of asset k at date t:

$$q_k^t = \frac{\rho_k^t}{\sum_{i=1}^I w_i^t} = \sum_{i=1}^I \lambda_{i,k}^t r_i^t \tag{7}$$

The real payoff of asset k at time $t+1$ can be calculated as

$$R_k^{t+1} = \frac{A_k^{t+1}}{\sum_{l=1}^K A_l^{t+1}} \tag{8}$$

2.2 The wealth dynamics in discrete time

Apart from the choice of the investments and the market structure, we have to describe how the wealth of the investors is determined in period $t+1$. We investigate the case of investors who never consume, but reinvest their investment earnings completely. For simplicity, we assume that investors do not receive income from labor. Hence, we suppose that $w_i^{t+1} = D_i^{t+1}$ for all times t and agents i. We may rewrite the evolution of relative wealth as

$$\begin{aligned} r_i^{t+1} &= r_i^t \sum_k \frac{\lambda_{i,k}^t}{\sum_j \lambda_{j,k}^t r_j^t} R_k^{t+1} \\ &= r_i^t + r_i^t \left(\sum_{k=1}^K R_k^{t+1} \frac{\lambda_{i,k}^t}{\sum_{j=1}^I \lambda_{j,k}^t r_j^t} - 1 \right). \end{aligned} \tag{9}$$

We will study the case in which the trading strategies $\lambda_{i,k}^t = \lambda_{i,k}$ do not depend on time. Hence, we will drop the index t. In this case, the wealth dynamic is only triggered by the random payments. We will always stick to the following assumption.

Assumption 2.1. *All agents invest a strictly positive amount into any asset, i.e. the values $\lambda_{i,k}$ are strictly positive. In economic terms, all agents are completely diversified.*

3 The wealth dynamics in continuous time

3.1 A continuous time approximation

In the current section we describe how a continuous time approximation of the evolutionary model can be constructed. Assuming that dividends are paid at a higher frequency, we state precise conditions for the convergence of the discrete time wealth process to a continuous time limit. It turns out that the limiting process can be characterized as the solution of an integral equation in a random environment.

The functional limit theorem and technical conditions under which we obtain convergence are described in the current section. The next section provides an economic foundation. Our approximation results bridge the gap between the evolutionary approach and the theory of continuous time processes which are commonly used in mathematical finance.

Let us now turn to the construction of the continuous-time approximation. Given $n \in \mathbb{N}$, we let a new time grid be given by the time points $\left\{ l \cdot n^{-1} : l \in \mathbb{N}_0 \right\}$. Dividends are paid at these dates, and the corresponding dividend process is a discrete time stochastic process denoted by $(A^{(n),s/n})_{s \in \mathbb{N}_0}$. By convention, we fix $A^{(n),0} = a_0 \in \mathbb{R}^K_+$.

Assumption 3.1. *For all $n \in \mathbb{N}$ and $s \in \mathbb{N}_0$, we suppose that with probability one*

$$\sum_{k=1}^{K} A_k^{(n),s/n} > 0.$$

Analogous to (8), the real returns of the assets are given by the expressions

$$R_k^{(n),s/n} = \frac{A_k^{(n),s/n}}{\sum_{l=1}^{K} A_l^{(n),s/n}}. \tag{10}$$

As before we suppose that trading takes place immediately after dividends have been received, but we will no longer assume that total wealth is invested. At times $0, \frac{1}{n}, \frac{2}{n}, \ldots$ agents invest only a fraction $\alpha^n \in (0,1]$ of their wealth in the market. This assumption modifies the dynamics described by equation (9). For fixed n, the wealth dynamics is now given by the following recursive scheme

$$r_i^{(n)}(t_{n,l+1}) = (1-\alpha^n) \cdot r_i^{(n)}(t_{n,l}) + \alpha^n \cdot r_i^{(n)}(t_{n,l}) \sum_{k=1}^{K} \frac{\lambda_{i,k} R_k^{(n),t_{n,l+1}}}{\sum_{j=1}^{I} r_j^{(n)}(t_{n,l}) \lambda_{j,k}}, \tag{11}$$

where $t_{n,l} = \frac{l}{n}$ and $r_0^{(n)} = r_0 \in \Delta_I$. Here, Δ_I denotes the simplex in $\mathbb{R}^I$.

We are interested in a continuous time approximation for $n \to \infty$ where we choose $\alpha^n = \frac{1}{n}$. For this purpose, it is convenient to extend all discrete time processes to continuous time. The continuous time extension of real returns $R^{(n)}$ is defined by the piecewise constant process

$$R^{(n)} := R^{(n),0} \cdot \mathbf{1}_{\{0\}} + \sum_{s=0}^{\infty} R^{(n),(s+1)/n} \cdot \mathbf{1}_{\left(\frac{s}{n}, \frac{s+1}{n}\right]}. \tag{12}$$

The wealth process $r^{(n)}$ is extended to continuous time by linear interpolation. For $t_{n,l} \le s \le t_{n,l+1}$ and $i = 1, 2, \ldots, I$, we let

$$\begin{aligned} r_i^{(n)}(s) \quad &:= \quad r_i^{(n)}(t_{n,l}) + n(s - t_{n,l}) \left(r_i^{(n)}(t_{n,l+1}) - r_i^{(n)}(t_{n,l}) \right) \\ &= \quad r_i^{(n)}(t_{n,l}) \\ &\quad + \int_{t_{n,l}}^{s} r_i^{(n)}(t_{n,l}) \left(\sum_{k=1}^{K} \frac{\lambda_{i,k} R_k^{(n),t_{n,l+1}}}{\sum_{j=1}^{I} r_j^{(n)}(t_{n,l}) \lambda_{j,k}} - 1 \right) du \end{aligned} \tag{13}$$

We will provide precise conditions under which the wealth processes $r^{(n)}$ converge to a continuous time limit r as $n \to \infty$. The limiting process r is characterized as the pathwise solution of an integral equation in a random environment. In the next proposition, we investigate the relevant family of integral equations. Under weak conditions, these possess a unique continuous solution.

Proposition 3.2. *Let Δ_I and Δ_K denote the simplices in $\mathbb{R}^I$ and $\mathbb{R}^K$, respectively. Let $T : \mathbb{R}_+ \to \Delta_K$ be measurable. Assume that $r_0 \in \Delta_I$. Then the coupled integral equations*

$$r_i(s) \quad = \quad r_{i,0} + \int_0^s r_i(s') \left(\sum_{k=1}^{K} \frac{\lambda_{i,k} T_k^{s'}}{\sum_{j=1}^{I} r_j(s') \lambda_{j,k}} - 1 \right) ds', \tag{14}$$

with $i = 1, 2, \ldots, I$, possess a unique continuous solution $r : \mathbb{R}_+ \to \Delta_I$.

Proof. See appendix. □

The existence of a continuous-time limit of the evolutionary stock market model relies on appropriate conditions on the real return processes. Key to the analysis is the following technical theorem which provides bounds on the pathwise approximation error. Economic conditions on the dividend processes guarantee that these errors are asymptotically zero, cf. Section 3.2

Theorem 3.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. For each $n \in \mathbb{N}$, we let $(R^{(n),(s-1)/n})_{s\in\mathbb{N}}$ be a sequence of random variables on Ω with values in Δ_K. $R^{(n)}$ is extended to a continuous time process by (12). Assume that $r^{(n)}$ is defined according to (11) and (13) with $r^{(n)}(0) = r_0 \in \Delta_I$.*

Let $(T^s)_{s\in\mathbb{R}_+}$ be a stochastic process on Ω with values in Δ_K that is jointly measurable in $\omega \in \Omega$ and $s \in \mathbb{R}_+$. Suppose that r is the pathwise unique continuous solution of (14).

Then there exists for every $t \geq 0$ a non-random constant D such that for all $n \in \mathbb{N}$ the following inequality holds:

$$\sup_{0\leq s\leq t} \|r(s) - r^{(n)}(s)\|_{\mathbb{R}^I} \leq D \cdot \left(\frac{1}{n} + \int_0^{t+\frac{1}{n}} \|T^u - R^{(n),u}\|_{\mathbb{R}^K}\, du\right), \tag{15}$$

where $\|\cdot\|_{\mathbb{R}^I}$ and $\|\cdot\|_{\mathbb{R}^K}$ are given norms on $\mathbb{R}^I$ and $\mathbb{R}^K$, respectively.

Proof. See appendix. □

As a consequence of the last theorem we obtain convergence of the discrete-time wealth processes to a continuous-time wealth processes, if the right hand side of (15) converges to zero as $n \to \infty$. For different modes of convergence, this fact is stated rigorously in the next corollary.

Corollary 3.4. *Consider the same setting as in Theorem 3.3. Let*

$$Y_t^n := \int_0^t \|T^u - R^{(n),u}\|_{\mathbb{R}^K}\, du\,. \tag{16}$$

Then the following implications hold:

(1) If Y_t^n converges for all $t \in \mathbb{R}_+$ to 0 almost surely, then $r^{(n)}$ converges to r uniformly on compacts with probability 1.

(2) If Y_t^n converges for all $t \in \mathbb{R}_+$ to 0 in probability, then $r^{(n)}$ converges to r uniformly on compacts in probability and in L^p for $p \in [1, \infty)$.

(3) If Y_t^n converges for all $t \in \mathbb{R}_+$ to 0 in L^∞, then $r^{(n)}$ converges to r uniformly on compacts in probability and in L^p for $p \in [1, \infty]$.

Proof. See appendix. □

3.2 Dividend processes in continuous time

We fix a probability space $(\Omega, \mathcal{F}, P)$, and assume that all random variables and processes are defined on Ω. We suppose now that the dividend payments $(A^{(n),s/n})_{s\in\mathbb{N}_0}$ ($n \in \mathbb{N}$) are driven by value processes earned by firms. More specifically, let $S^t \in \mathbb{R}_+^K$ be the stochastic process of the excess value generated by K firms corresponding to the assets $k = 1, 2, \ldots, K$, i.e. the process of cumulated dividends. We make the following assumption:

Assumption 3.5. *The value process S^t is cadlag and strictly increasing in the following sense: for given $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $\epsilon > 0$ it holds that*

- $\forall k$: $S_k^{t+\epsilon}(\omega) - S_k^t(\omega) \geq 0$
- $\exists k$: $S_k^{t+\epsilon}(\omega) - S_k^t(\omega) > 0$

We will assume that the value process S^t is related to the dividend payments in the following way:

(1) $A^0 = S^0$,

(2) $A^t = S^t - S^{t-1}$ for $t \in \mathbb{N}$.

In other words, at time t the firm pays the complete incremental value generated between times $t-1$ and t as dividends to the investors. The real payoff of asset k at time $t+1$ can therefore be calculated as

$$R_k^{t+1} = \frac{A_k^{t+1}}{\sum_{l=1}^K A_l^{t+1}} = \frac{S_k^{t+1} - S_k^t}{\sum_{l=1}^K (S_l^{t+1} - S_l^t)}. \tag{17}$$

In the continuous time approximation, dividends are paid at a higher frequency. Given $n \in \mathbb{N}$, we define on the new time grid $\frac{1}{n}\mathbb{N}$ for $s \in \mathbb{N}_0$

(1) $A^{(n),0} = S^0$,

(2) $A^{(n),(s+1)/n} = S^{(s+1)/n} - S^{s/n}$.

In terms of the value process, real returns are thus given by

$$R_k^{(n),(s+1)/n} = \frac{A_k^{(n),(s+1)/n}}{\sum_{l=1}^K A_l^{(n),(s+1)/n}} = \frac{S_k^{(s+1)/n} - S_k^{s/n}}{\sum_{l=1}^K (S_l^{(s+1)/n} - S_l^{s/n})}. \tag{18}$$

At time 0, we obtain returns not depending on n:

$$R_k^{(n),0} = \frac{A_k^{(n),0}}{\sum_{l=1}^K A_l^{(n),0}} = \frac{S_k^0}{\sum_{l=1}^K S_l^0}. \tag{19}$$

We extend again $R^{(n)}$ to a continuous time process $(R^{(n)}(\omega, u))_{u \geq 0}$ by formula (12). If the real dividends $R^{(n)}$ converge in an appropriate sense to a limiting process T, we can apply Theorem 3.3 and Corollary 3.4 to obtain a continuous time approximation of the wealth process. We are thus interested in the question when the stochastic processes $R^{(n)}$ converge to a limiting process T and how this process is related to the firms' value process S. For this purpose, it is very helpful to establish a representation of S is terms of locally finite kernels.

A representation of the firm value process. By Assumption 3.5, for $\omega \in \Omega$ the components $S_k(\omega)$ $(k = 1, \ldots, K)$ of the firm value process are cumulative distribution functions of a positive locally finite Borel measure $\mu_k(\omega)$ on $\mathbb{R}_+$. More precisely, μ_k $(k = 1, 2, \ldots, K)$ is a locally finite kernel from Ω to $\mathbb{R}_+$. Here, a mapping $\mu : \Omega \times \mathcal{B}(\mathbb{R}_+) \to \bar{\mathbb{R}}_+$ is called a locally finite kernel, if

(1) $\mu(\cdot, B) : \Omega \mapsto \bar{\mathbb{R}}_+$ is measurable;

(2) $\mu(\omega, \cdot)$ is a locally finite measure on $\mathbb{R}_+$ for all $\omega \in \Omega$,

where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel-σ-algebra on $\mathbb{R}_+$.

Given a probability measure P on $(\Omega, \mathcal{F})$, every locally finite kernel μ from Ω to $\mathbb{R}_+$ induces a unique σ-finite measure $P\mu$ on $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$. , cf. Proposition A.2. $P\mu$ is uniquely defined by setting

$$P\mu(A \times B) := \int_A \mu(\omega, B)\, dP \qquad (A \in \mathcal{F},\ B \in \mathcal{B}(\mathbb{R}_+))\,. \tag{20}$$

The following theorem provides a canonical representation of the firm value process S which is useful when investigating the convergence to a continuous time dividend process. The notion of *exhausting sequence* is given in Definition A.1 in the appendix.

Theorem 3.6. *Suppose that Assumption 3.5 holds. Then we find a canonical representation of S^t in terms of a locally finite kernel μ from Ω to $\mathbb{R}^+$ and measurable functions $f_k : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the sum of the functions $(\omega, u) \mapsto f_k(\omega, u)$, $k = 1, \dots, K$, is $P\mu$-almost everywhere positive. Namely, for every $1 \leq k \leq K$ and for every $t \geq 0$ the firm value process $S : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^K$ satisfies for P-almost all $\omega \in \Omega$,*

$$S_k^t(\omega) = \int \mathbf{1}_{[0,t]}(u)\ f_k(\omega, u)\ \mu(\omega, du). \tag{21}$$

For every exhausting sequence (C_N) for P and μ, the functions $f 1_{C_N}$ are $P\mu$-integrable .

Proof. See appendix. □

Convergence to a continuous time dividend process. Assumption 3.5 implies that the discrete-time real return processes converge to a continuous-time limit. The proof is based on a martingale argument which can be found in the appendix.

Proposition 3.7. *Suppose that Assumption 3.5 holds. We suppose that S^t is represented according to (21). Then, for $P\mu$-almost all (ω, u), the limit of $R^{(n)}(\omega, u)$ exists for $n \to \infty$ and equals*

$$\lim_{n \to \infty} R_k^{(n)}(\omega, u) = \frac{f_k(\omega, u)}{\sum_{l=1}^K f_l(\omega, u)}. \tag{22}$$

Proof. See appendix. □

Dividend convergence and Euler approximation. In this paragraph we provide sufficient conditions on the firms' value process S^t which ensure the convergence of the discrete time wealth processes $r^{(n)}$ to a continuous time process r. In terms of the family of random variables Y_t^n $(t \in \mathbb{R}_+,\ n \in \mathbb{N})$ conditions have been derived in Section 3.1, see in particular Corollary 3.4. We will now combine these results with representation (21) of Theorem 3.6.

If the measure $\mu(\omega, \cdot)$ dominates the Lebesgue measure for P-almost all $\omega \in \Omega$, strong implications can be derived. In this case, with probability 1 the limiting statement (22) holds both μ- and Lebesgue-almost everywhere, and we obtain the following Euler approximation.

Corollary 3.8. *Suppose that the Assumption 3.5 holds, and let a representation of the value process S be given according to Theorem 3.6. Suppose that P-almost surely μ dominates the Lebesgue measure. For $k = 1, \dots, K$, we set $g_k = f_k$ if $\sum_{l=1}^K f_l > 0$, and $g_k = 1$ else. Define the process $T = g_k \cdot \left(\sum_{l=1}^K g_l\right)^{-1}$. Then $r^{(n)}$ converges to r defined in (14) uniformly on compacts with probability* 1.

Proof. See appendix. □

The condition on Corollary 3.8 is not always satisfied. Given a value process S, we can in general not expect to find a representation (21) such that μ dominates the Lebesgue measure as the next example shows.

Example 3.9. *For the construction of the counterexample we may w.l.o.g. focus on the deterministic case. Let* $K = 1$, *and define* $\nu := \sum_{l \in \mathbb{N}} \frac{1}{2^l} \delta_{q_l}$, *where* $q : \mathbb{N} \to \mathbb{Q}$, $l \mapsto q_l$ *is a bijection. Assume that* $S : \mathbb{R}_+ \to \mathbb{R}_+$ *is given by* $S^t = \nu([0,t])$. *It is not possible to find a measure* μ *dominating the Lebesgue measure and a density* f *which is* μ*-almost surely positive such that* S *can be represented by* $S^t = \int \mathbf{1}_{[0,t]} f d\mu$.

In terms of the representing kernel μ in (21), Corollary 3.8 provides a sufficient condition on the firms' value process S^t which ensures the convergence of the discrete time wealth processes $r^{(n)}$ to a continuous time process r. The next proposition and Corollary 3.11 give a condition in terms of the representing densities f_k $(k = 1, \dots, K)$. If these functions are sufficiently regular, then the continuous time approximation of the wealth process is valid – irrespectively of the properties of the representing kernel μ.

Proposition 3.10. *Suppose that Assumption 3.5 holds. Assume that there exists a canonical representation according to Theorem 3.6 such that the mappings* $f_k(\omega, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *are Lebesgue-almost everywhere continuous for* $1 \le k \le K$ *with probability one. Then for* P*-almost all* $\omega \in \Omega$ *the sum of the functions* $(f_k(\omega))_{k=1,\dots,K}$ *is Lebesgue-almost everywhere positive and the limit of* $R^{(n)}(\omega)$ *exists Lebesgue-almost everywhere and equals*

$$\lim_{n \to \infty} R_k^{(n)}(\omega) = \frac{f_k(\omega)}{\sum_{l=1}^K f_l(\omega)}. \tag{23}$$

Proof. See appendix. □

Corollary 3.11. *Suppose that the assumptions of Proposition 3.10 are satisfied. For* $k = 1, \dots, K$, *we set* $g_k = f_k$ *if* $\sum_{l=1}^K f_l > 0$, *and* $g_k = 1$ *else. Define the process* $T = g_k \cdot \left(\sum_{l=1}^K g_l \right)^{-1}$. *Then* $r^{(n)}$ *converges to* r *defined in (14) uniformly on compacts with probability* 1.

Proof. See appendix. □

4 Deterministic dynamics

The continuous time wealth dynamics (14) is driven by the relative dividend process T. We are interested in the relative performance of the strategies which is characterized by the asymptotic behavior of the wealth process as $t \to \infty$. In this section we will focus on a special case – assuming that T is deterministic and constant which corresponds to no dividend risk. While fundamentals are fixed, prices and wealth vary due to market interaction.

The wealth dynamics in the absence of fundamental risk is described by an autonomous differential equation. We will analyze its asymptotics employing the technique of Lyapunov functions. The analysis forms also the basis for the investigation of the more complex situation with a stochastic dividend process. This case will be investigated in Buchmann & Weber (2005).

4.1 The semiflow of the wealth dynamics

We suppose $T \equiv \pi$ for fixed $\pi \in \Delta_K$. For the whole section we make the following assumption.

Assumption 4.1. *The real dividends π are strictly positive, i.e. $\pi_k > 0$ for $1 \leq k \leq K$.*

Define a mapping $N : \Delta_I \to \mathbb{R}^I$ by

$$N_i(r) \quad = \quad \sum_{k=1}^{K} \frac{\pi_k \lambda_{i,k}}{\sum_{j=1}^{I} r_j \lambda_{j,k}} - 1. \tag{24}$$

Moreover, let the vector field $\psi : \Delta_I \to \mathbb{R}^I$ be given by

$$\psi_i(r) \quad = \quad r_i \cdot N_i(r). \tag{25}$$

Then the integral equation (14) reduces to an autonomous differential equation, namely

$$\dot{r}(t) \quad = \quad \psi(r), \qquad r(0) = r_0. \tag{26}$$

This ordinary differential equation describes the wealth dynamics in continuous time.

The ordinary differential equation (26) can be extended to an open neighborhood of the simplex Δ_I. Namely, since the linear mappings $r \mapsto \sum_{j=1}^{I} r_j \lambda_{j,k}$ in the denominator of (24) are continuous on $\mathbb{R}^I$ and strictly positive on Δ_I, N and ψ in (24) and (25) are defined on an open neighborhood D of Δ_I. Then, for given initial value $r_0 \in D$, the solution of (26) exists for all times t smaller than some maximal $t^+(r_0) > 0$ and larger than some minimal $t^-(r_0) < 0$.

We associate a flow

$$\phi : \Gamma \to D, \ (t, r_0) \mapsto \phi_t(r_0) \tag{27}$$

with the ordinary differential equation (26), where $\phi_t(r_0)$ is the value of the solution of (26) at time t when the initial value is r_0. Its domain $\Gamma \subseteq \mathbb{R} \times D$ is given by

$$\Gamma = \bigcup_{r \in D} (t^-(r), t^+(r)) \times \{r\} \ .$$

A flow satisfies the following four properties: (1) Γ is open in $\mathbb{R}_+ \times D$. (2) $\phi : \Gamma \to D$ is continuous. (3) $\phi_0 = id_D$. (4) For initial value $r \in D$ and times $s \in (t^-(r), t^+(r))$, $t \in (t^-(\phi_s(r)), \phi_s(t^+(r)))$, it holds that $t^-(r) < s + t < t^+(r)$ and $\phi_t(\phi_s(r)) = \phi_{s+t}(r)$.

We need some concepts from convex geometry. The *relative interior* of a convex set C will be denoted by ri (C), i.e.

$$\text{ri } (C) = \{c \in C : \ \exists \epsilon > 0 \ \forall y \in C \quad \forall |\delta| < \epsilon \ \ c + \delta(y - c) \in C\} \ .$$

The *relative boundary* of a convex set C is defined by $\partial^*(C) := \bar{C} \backslash \text{ri } (C)$.

In contrast to the standard topological concept of open sets, the set ri (C) is never empty, whenever the convex set C is not empty. For instance, the set $C = \{x\}$ has relative interior ri $(C) = \{x\}$.

Definition 4.2. *A set $M \subseteq D$ is called* invariant*, if $\phi_t(r) \in M$ for all $r \in M$ and $t \in (t^-(r), t^+(r))$. M is called* positively invariant*, if $\phi_t(r) \in M$ for all $r \in M$ and $t \in [0, t^+(r))$.*

It is not difficult to show that Δ_I is invariant, cf. Amann (1983), Corollary 16.10. This has implications for the domain Γ of the flow. Since Δ_I is compact and invariant, the solution of the differential equation (26) exists for all times $t \in \mathbb{R}$, if the initial value $r_0 \in \Delta_I$ ([(Amann 1983)], Remark (17.3)). We obtain that

$$\Gamma \ = \ \Big(\mathbb{R} \times \Delta_I\Big) \ \cup \ \left(\bigcup_{r \in D \backslash \Delta_I} (t^-(r), t^+(r)) \times \{r\}\right). \tag{28}$$

Besides the simplex Δ_I also the sets $\partial^*(\Delta_I)$ and ri (Δ_I) are invariant; this is implied by standard arguments, cf. Amann (1983), Corollary 16.10. Moreover, the vertices e_i of the simplex Δ_I are fixed points of the flow. Here, e_i denotes the ith unit vector in $\mathbb{R}^I$.

Finally, we define for $J \subseteq I$ the subsimplices

$$\Delta_J := \left\{ \sum_{i \in J} r_i e_i : \ r \in \mathbb{R}_+, \quad \sum_{j \in J} r_j = 1 \right\}.$$

For $J \subseteq I$, $\Delta_J \subseteq \Delta_I$ is invariant. In economic terms, the restriction to a simplex Δ_J, $J \subseteq I$, $J \neq I$ corresponds to a smaller economy where only agents from set J are present. If the initial value is an element of the boundary, i.e. $r \in \partial^*(\Delta_I)$, the wealth dynamics is effectively of lower dimension. Hence, we need to analyze the dynamics for initial values $r \in ri(\Delta_I)$.

4.2 A Lyapunov function and LaSalle's criterion

We will now characterize the asymptotic behavior of the semiflow of the wealth dynamics. For this purpose, we will investigate a Lyapunov function of the flow. Lyapunov functions are defined in terms of derivatives along the orbit of the flow inside a given set M, cf. Amann (1983). We do not need this definition in full generality. Instead we will work with the following sufficient criterion that characterizes Lyapunov functions on an open neighborhood of M by their gradient.

Lemma 4.3. *Let $M \subseteq D$. A differentiable function $\Phi : U \to \mathbb{R}$, defined on some open neighborhood U of M, is a Lyapunov function on M of the semiflow ϕ associated with ψ if*

$$\dot{\Phi}(r) := \nabla_r \Phi(r) \psi(r) \leq 0 \qquad \forall r \in M.$$

Φ is non-increasing along trajectories $\phi_t(r_0)$ for $r_0 \in M$. We recall the following corollary of the invariance principle of LaSalle.

Corollary 4.4. *Let $M \subseteq D$ be closed and positively invariant for the semiflow ϕ. Assume that Φ is a Lyapunov function on M. Let M_Φ be the largest invariant subset of*

$$\left\{ r \in M : \dot{\Phi}(r) = 0 \right\}.$$

Then, M_Φ attracts all points of M, i.e. for all $r \in M$ we have

$$\lim_{t \to t^+(r)} \text{dist } (\Phi_t(r), M_\Phi) \ = \ 0 \ .$$

We will next use Corollary 4.4 to characterize the minimal attractor of Δ_I. It describes the long-run wealth distribution in the economy, if initially no more than I agents are present. A more detailed analysis allows us to determine the minimal attractor of $ri(\Delta_I)$. This second attractor captures the long-run wealth distribution in the economy, if initially the wealth of *all* I investors is positive, i.e. if initially (and thus for every finite time) exactly I agents are present.

A Lyapunov function Φ for the flow that describes the wealth dynamics is given in the following lemma.

Lemma 4.5. *Suppose that Assumptions 2.1 and 4.1 are satisfied. The function $\Phi : D \to \mathbb{R}$, defined as*

$$\Phi(r) := -\sum_{k=1}^{K} \pi_k \log(\sum_{j=1}^{I} \lambda_{j,k} r_j) + \sum_{k=1}^{K} \sum_{j=1}^{I} \lambda_{j,k} r_j \ , \tag{29}$$

is a Lyapunov function for the flow ϕ on Δ_I and satisfies on D the equation $N = -\nabla_r \Phi$. The Lyapunov function Φ is convex on Δ_I.

Proof. See appendix. □

The next corollary completely characterizes the long-run wealth distributions in an economy with no more than I agents.

Corollary 4.6. *Suppose that Assumptions 2.1 and 4.1 are satisfied. The minimal attractor of Δ_I for the flow ϕ is given by*

$$\mathcal{A} := \left\{ r \in \Delta_I : \sum_{i=1}^{I} r_i N_i^2(r) = 0 \right\}.$$

$\mathcal{A}$ is a set of fix points. In particular, for all $r \in \Delta_I$ the ω-limit set $\omega(r)$ is included in $\mathcal{A}$.

Proof. See appendix. □

4.3 The global attractor

We will now investigate the asymptotic properties of the solution of the ordinary differential equation (26) for initial values $r(0) = r_0 \in ri(\Delta_I)$. Recall that the differential equation (26) describes the dynamics of investors' wealth. We are interested in the smallest closed set $\mathcal{B}$ attracting all points $r \in \text{ri}\ (\Delta_I)$ which is given by

$$\mathcal{B} = \overline{\bigcup_{r \in \text{ri}\ (\Delta_I)} \omega(r)}.$$

$\mathcal{B}$ characterizes the long-run wealth distributions in an economy with I agents, if initial wealth of *all* investors is positive. The analysis thus refines Corollary 4.6 in which we determined the long-run wealth asymptotics in an economy with *no more* than I agents.

The minimal attractor $\mathcal{B}$ of the relative interior $ri(\Delta_I)$, the attractor $\mathcal{A}$ of the whole simplex Δ_I, and the minima of the Lyapunov function Φ are closely related. We denote the set of global minima of the Lyapunov function Φ on Δ_I by $\mathcal{A}_{\min}$. Since Φ is a convex function, global and local minima coincide.

Remark 4.7. *Elementary relations between the attractors of the simplex and its relative interior and the minima of the Lyapunov function are described in Proposition A.3 in Section A.4. In particular, the following holds.*

- *$\mathcal{A}_{\min}$ is a non empty, closed, convex set of fixed points for Φ.*
- *Both $\mathcal{B}$ and $\mathcal{A}_{\min}$ are subsets of $\mathcal{A}$.*
- *Finally, $\mathcal{A}_{\min}$ is a subset of $\mathcal{B}$, if $\mathcal{A}_{\min}$ contains points of the relative interior $ri\ (\Delta_I)$.*

In certain cases the minimal attractor $\mathcal{B}$ of the relative interior of the simplex can completely be characterized by the minima of the Lyapunov function. In this case, these minima determine the long-run wealth distributions of the dynamics. The next theorem provides conditions.

Theorem 4.8. *Suppose that Assumptions 2.1 and 4.1 are satisfied.*

Assume that one of the following two conditions is satisfied.

(1) Φ is strictly convex on the boundary $\partial^*(\Delta_I)$, that is $\Phi : D \to \mathbb{R}$ is strictly convex for all convex subsets of the boundary $\partial^*(\Delta_I)$.

(2) $\Phi(e_i) = \min_{g\in\partial^*(\Delta_I)} \Phi(g)$ for some $i \in I$.

Then $\mathcal{B} \subseteq \mathcal{A}_{\min}$. If additionally $\mathcal{A}_{\min}$ contains points of the relative interior of Δ_I, then $\mathcal{B} = \mathcal{A}_{\min}$.

Proof. See appendix. □

In terms of the convexity of the Lyapunov function, condition (1) formalizes a sufficient condition for a characterization of the asymptotics via the minima of Φ. The next proposition further investigates hypothesis (1) of Theorem 4.8. It is shown that (1) is satisfied if the dimension of the hyperspace defined by the trading strategies is large compared to the number of investors.

For this purpose, we define a function

$$\tilde{\Phi} : \begin{cases} (\mathbb{R}_+ \setminus \{0\})^K & \to \quad \mathbb{R} \\ \quad x & \mapsto \quad -\sum_{k=1}^K \pi_k \log(x_k) + \sum_{k=1}^K x_k. \end{cases} \tag{30}$$

The Lyapunov function Φ can be recovered from $\tilde{\Phi}$ by

$$\Phi(r) = \tilde{\Phi}\left(\left(\sum_{i=1}^I r_i \lambda_{i,k} \right)_k \right) \tag{31}$$

Recall that by (7) the argument of $\tilde{\Phi}$ equals the real price vector $(q_k)_{k=1,2,\ldots,K}$ of the assets.

The minimization of the Lyapunov function Φ on the space of wealth distributions consists thus of the two steps: minimize firstly the associated Lyapunov function $\tilde{\Phi}$ on the price space, and find secondly the wealth distributions that support this price vector given the fixed strategy profile.

Next, define the matrix

$$M^{(i)} = (\lambda_j - \lambda_i)_{j\in I\setminus\{i\}} \in \mathbb{R}^{K,I-1} \qquad (i = 1, 2, \ldots, I). \tag{32}$$

$M^{(i)}$ defines a linear mapping from $\mathbb{R}^{I-1}$ to $\mathbb{R}^K$, and we denote its nullspace by $\ker M^{(i)}$ and its rank by rg $M^{(i)}$. The rank rg $M^{(i)}$ does not depend on the choice of i. It equals the dimension of the minimal affine hyperspace that contains the trading strategies.

Proposition 4.9. *Let $d = I - 1 - rg\ M^{(i)} = \dim(\ker M^{(i)})$.*

(1) If $d = 0$, then Φ is strictly convex. In particular, Φ is strictly convex on the boundary $\partial^*(\Delta_I)$.

(2) If $d \geq 2$, then Φ is not strictly convex on the boundary $\partial^*(\Delta_I)$.

(3) If $d = 1$, then Φ is generically strictly convex on the boundary $\partial^*(\Delta_I)$. To be more precise, set $G := \{0, e_1, e_2, \ldots, e_I\} \setminus \{e_i\}$. If $d = 1$, then Φ is strictly convex on the boundary, if and only if for all $v \in G$,

$$\ker M^{(i)} \cap span\{u - \bar{u} : \ u, \bar{u} \in G \setminus \{v\}\} = \{0\}.$$

Proof. See appendix. □

Recall that $\mathcal{B} \subseteq \mathcal{A}_{\min}$, if Φ is strictly convex on the boundary $\partial^*(\Delta_I)$. Proposition 4.9 provides conditions. The analysis of the attractor $\mathcal{B}$ of the relative interior of the simplex Δ_I is more complicated, if $\mathcal{B} \not\subseteq \mathcal{A}_{\min}$. For details we refer to the appendix, cf. Section A.6.

4.4 The minima of the Lyapunov function

We have seen under which condition the long run asymptotics can be understood in terms of the minima of the Lyapunov functions. The aim of the current section is twofold: first, we will further investigate the structure of the minima; second, we will discuss implications for the long run behavior of the wealth process.

In (31) we have already seen that the Lyapunov function Φ is the composition of a strictly convex function $\tilde{\Phi}$ and a linear mapping. Thus, we investigate the minima $\mathcal{A}_{\min}$ of Φ in two steps. First we determine the minima of $\tilde{\Phi}$. Then, we investigate the implications for the minima of Φ.

Lemma 4.10. *Suppose that Assumptions 2.1 and 4.1 are satisfied.*

(1) π is the unconstraint absolute minimizer of $\tilde{\Phi}$.

(2) We denote by $\Lambda \subseteq \Delta_K$ the convex hull of the trading strategies $\lambda_1, \ldots, \lambda_I$. Then there exists a unique x^* such that

$$\tilde{\Phi}(x^*) = \inf_{x \in \Lambda} \tilde{\Phi}(x). \tag{33}$$

The minimizer x^* depends on both the real dividends π and the polyhedral set Λ. $x^* = \pi$, if and only if $\pi \in \Lambda$.

Proof. See appendix. □

Given the minimizer x^* of $\tilde{\Phi}$ on Λ, the set of minima $\mathcal{A}_{\min}$ of the Lyapunov function Φ on the simplex Δ_I is essentially determined by the solution of a linear equation. $\mathcal{A}_{\min}$ is a polyhedral set, that is, the convex hull of finitely many points.

Lemma 4.11. *$\mathcal{A}_{\min}$ is a non empty polyhedral set and can be represented by*

$$\mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{i=1}^{I} r_i \lambda_{i,k} = x_k^* \quad \text{for all } 1 \leq k \leq K \right\}. \tag{34}$$

Proof. See appendix. □

Certain situations are particularly easy to analyze. These include those cases in which the minimizer x^* equals a trading strategy λ_i for some $i \in I$. We will investigate the asymptotics for these special cases. Before we turn to this point, the following proposition formulates necessary and sufficient conditions.

Proposition 4.12. *Let $i \in I$. Then the following conditions are equivalent.*

(i) $\lambda_i = x^*$.

(ii) $\nabla \tilde{\Phi}(\lambda_i) \cdot (\lambda_j - \lambda_i) \geq 0$ *for all* $j \in I \setminus \{i\}$.

(iii) $\sum_{k=1}^{K} \pi_k \frac{\lambda_{j,k}}{\lambda_{i,k}} \leq 1$ *for all* $j \in I \setminus \{i\}$.

Proof. See appendix. □

Let uns finally discuss the long run behavior of the model. If one of the conditions of the last proposition is satisfied, the asymptotics of the model can easily be characterized. Then, the global attractor $\mathcal{B}$ is a subset of the the minima $\mathcal{A}_{\min}$.

Corollary 4.13. *Assume that one and thus all of the equivalent conditions in Proposition 4.12 hold. Then*

$$\mathcal{B} \subseteq \mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{j=1}^{I} r_j \lambda_j = \lambda_i \right\}. \tag{35}$$

If additionally $\mathcal{A}_{\min}$ contains a point in the relative interior ri (Δ_I), then $\mathcal{B} = \mathcal{A}_{\min}$ in (35).

Proof. See appendix. □

Corollary 4.14. *Assume that one and thus all of the equivalent conditions in Proposition 4.12 hold. If λ_i is an extremal point Λ, then*

$$\mathcal{B} = \mathcal{A}_{\min} = \{e_i\}.$$

Proof. See appendix. □

Example 4.15. *Let us consider a special case. Assume that $\lambda_i = \pi$, $\lambda_j \neq \pi$ for $j \neq i$. The strategy π is closely related to "betting your beliefs" as introduced by Breiman (1961). If λ_i is extremal in Λ, then agent i will asymptotically own total wealth, while all other agents loose everything. In particular, the implication holds, if all trading strategies are extremal points in Λ, i.e. if there are* no redundant strategies *present in the market. Then $\lambda_i = \pi$ is the unique global attractor. This parallels the results of Evstigneev et al. (2002) where an analogous strategy is characterized as the global attractor in a discrete-time model.*

In the preceding corollary it was assumed that some trading strategy is equal to the minimizer x^*. This hypothesis is, of course, not always satisfied. In general, the minimizer x^* of $\tilde{\Phi}$ is a convex combination of the trading strategy λ_j, $j \in I$. This convex combination involves some subset of the trading strategies, but possibly not all of them. The next proposition characterizes trading strategies which will never contribute to x^*. The consequences of the long run of the wealth process are discussed afterwards.

Proposition 4.16. *Assume that for some $i \in I$ the following inequality is satisfied, namely*

$$\sum_{k=1}^{K} \pi_k \frac{\lambda_{i,k}}{x_k^*} \neq 1. \tag{36}$$

If $\sum_{j=1}^{I} r_j \lambda_j = x^*$ or, equivalently, $r \in \mathcal{A}_{\min}$ for some $r \in \Delta_I$, then $r_i = 0$.

Proof. See appendix. □

Suppose now that r_0 is an initial value of the wealth distribution among investors with asymptotics $\omega(r_0) \subseteq \mathcal{A}_{\min}$. If the condition (36) of the preceding proposition is satisfied, then strategy λ_i dies out in the long run, that is, $r_i = 0$ for $r \in \omega(r_0)$. Condition (36) depends on Λ: whether a trading strategy dies out or not for initial value r_0 with $\omega(r_0) \subseteq \mathcal{A}_{\min}$, is determined by its business environment of competing trading strategies.

4.5 A rational benchmark

The vector π is the unconstraint minimizer of the function $\tilde{\Phi}$. This implies that whenever $\pi \in \Lambda$, then

$$\mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{j=1}^{I} r_j \lambda_{j,k} = \pi_k \text{ for all } 1 \leq k \leq K \right\}.$$

By (7) the vector π equals the price vector $(q_k)_{k=1,2,\dots,K}$ for any wealth distribution $r \in \mathcal{A}_{\min}$ and the given profile of trading strategies. Under conditions which we already discussed in previous sections the long-run wealth distributions $\mathcal{B}$ are characterized by $\mathcal{A}_{\min}$.

In this section we will compare our results to a rational benchmark of maximizing investors who are price takers in the Walrasian market. In contrast to the evolutionary perspective agents can now choose their trading strategies. It turns out that also in this context the vector π plays a special role.

We consider myopic agents who are price takers in a continuous time Walrasian market. The aim of the agents is to maximize the instantaneous gain or growth of their portfolio. By (26) and (24) the objective function of the agents $i = 1, 2, \dots, I$ is thus equal to

$$V_i^q : \begin{cases} \Delta_K & \rightarrow \quad \bar{\mathbb{R}}_+ \\ \lambda_i & \mapsto \quad \sum_{k=1}^K \frac{\pi_k}{q_k} \lambda_{i,k} - 1 \end{cases}$$

Here, $(q_k)_{k=1,2,\dots,K}$ equals by (7) the real price vector of the assets. In terms of the price, the market clearing conditions can be rewritten as

$$q_k = \sum_{i=1}^{I} \lambda_{i,k} \cdot r_i, \qquad k = 1, 2, \dots, K. \tag{37}$$

Under these conditions we obtain the following result.

Proposition 4.17. *The set of Walrasian equlibria in the economy of price taking myopic investors is given by*

$$\mathcal{E} = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_I) \in (\Delta_K)^I : \sum_{i=1}^{I} \lambda_{i,k} r_i = \pi_k \text{ for all } 1 \le k \le K \right\}.$$

In equilibrium the price vector q equals π. Moreover, in equilibrium the wealth vector $(r_i)_{i=1,2,\dots,I}$ of the investors is constant.

Proof. See appendix. □

Finally, observe that the set of Walrasian equilibria for given wealth vector $(r_i)_{i=1,2,\dots,I}$,

$$\mathcal{E} = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_I) \in (\Delta_K)^I : \sum_{j=1}^{I} \lambda_{j,k} r_j = \pi_k \text{ for all } 1 \le k \le K \right\},$$

and the set of minima of the Lyapunov function Φ,

$$\mathcal{A}_{\min} = \left\{ r \in \Delta_I : \sum_{j=1}^{I} r_j \lambda_{j,k} = \pi_k \text{ for all } 1 \le k \le K \right\},$$

for given strategy profile $(\lambda_i)_{i=1,2,\dots,I}$ with $\pi \in \Lambda$, are dual with respect to each other.

Remark 4.18. *Instead of price taking investors who maximize their objective functions V_i^q for given price vector q we could investigate an oligopolistic market game. In this situation the objective function of investors $i = 1, 2, \dots, I$ equals*

$$U_i(\lambda_1, \lambda_2, \dots, \lambda_I) = \sum_{k=1}^{K} \frac{\pi_k \lambda_{i,k}}{\sum_{j=1}^{I} r_j \lambda_{j,k}}.$$

In the Nash equilibrium of the strategic game, each investor $i \in I$ chooses her optimal λ_i given the trading strategies of the others. It can be shown that in this strategic situation the unique Nash equilibrium is the strategy profile $(\lambda_1, \lambda_2, \dots \lambda_I) = (\pi, \pi, \dots, \pi)$.

5 Conclusion

In this article, we have discussed a continuous time approximation for the evolutionary stock market model of Blume & Easley (1992). We provided conditions for the convergence of the Euler scheme to a nonlinear integral equation in a random environment. If dividend payments are increments of an excess value process of firms, the analysis reveals that a representation of the value process in terms of a locally finite kernel is useful. In particular, the Euler scheme converges, if the representing kernel dominates the Lebesgue measure, or – alternatively – if the representing densities are smooth enough.

For constant asset return, we investigate the long-run asymptotics of the continuous time wealth process.[1] In this case the integral equation reduces to an autonomous ordinary differential equation. The asymptotic behavior can be characterized by the minima of a Lyapunov function. This relationship was analyzed in detail in Section 4. In any non redundant market, the dividend vector π characterizes the unique surviving strategy (if present in the market) which consists in dividing income proportionally to relative payoffs of the assets.

Finally, we have investigated a rational benchmark. In the context of the evolutionary dynamics, the dividend vector π is closely related to both the absolute minimizers of the Lyapunov function and the superior strategy in any non redundant market. For rational investors, this dividend vector determines the set of Walrasian equilibria.

A Appendix

A.1 Proofs of Section 3.1

Proof of Proposition 3.2. Since all norms on finite dimensional vector spaces are equivalent, we do not have to specify a particular norm on $\mathbb{R}^I$ and $\mathbb{R}^K$, respectively. Of course, bounds and Lipschitz constants depend on the choice of the norms. For simplicity, we will denote the norms by $\|\cdot\|$.

The right hand side of the integral equation (14) depends on a function ψ with domain $\Delta_I \times \Delta_K$ defined by

$$\psi_i(r,T) = r_i\left(\sum_{k=1}^{K} \frac{\lambda_{i,k}T_k}{\sum_{j=1}^{I} r_j\lambda_{j,k}} - 1\right). \tag{38}$$

ψ is both bounded by some constant B and Lipschitz continuous with constant L as can be seen by the following arguments. First, ψ is affine in T. Second, observe that by assumption the trading strategies $\lambda_{j,k}$ are strictly positive. Hence, the zeros of the linear mapping

$$r \mapsto \sum_{j=1}^{I} r_j\lambda_{j,k}$$

are not contained in Δ_I. It follows that ψ is continuously differentiable on its compact domain, hence both bounded and Lipschitz continuous.

We first verify uniqueness. If r_1 and r_2 are two solutions, then by Lipschitz continuity of ψ we obtain

$$\sup_{0\le s\le t} \|r_1(s) - r_2(s)\| \le L\, t \sup_{0\le s\le t} \|r_1(s) - r_2(s)\|.$$

[1] The long-run asymptotics for stochastic dividend processes will be analyzed in future work, cf. Buchmann & Weber (2005).

This implies uniqueness for $t < 1/L$. A concatenation argument implies the identity $r_1(s) = r_2(s)$ for any $s \in \mathbb{R}_+$.

Next we prove existence. Define a sequence of functions $\rho^{(n)} : \mathbb{R}_+ \to \mathbb{R}^I$ by the following recursive scheme

$$\rho^{(n)}(t) \quad = \quad r_0 + \int_0^t \sum_{l=0}^{\infty} 1_{[\tau_{n,l},\tau_{n,l+1})}(u)\, \psi\left(\rho^{(n)}(\tau_{n,l}), \mathcal{T}_n^u\right) du\,,$$

where $\tau_{n,l} = l\, 2^{-n}$, $n \in \mathbb{N}$, $l \in \mathbb{N}_0$. Here, the second argument of ψ equals the average

$$\mathcal{T}_n^u := 2^n \cdot \int_{\tau_{n,l}}^{\tau_{n,l+1}} T^u du \qquad (u \in [\tau_{n,l}, \tau_{n,l+1})).$$

$\rho^{(n)}$ is continuous with values in Δ_I. As ψ is uniformly bounded, we obtain $\|\rho^{(n)}(t) - \rho^{(n)}(s)\| \leq B|t-s|$. Thus $K = \{\rho^{(n)} : \ n \in \mathbb{N}\}$ is relatively compact by the Theorem of Arćela-Ascoli. Hence, there exists a continuous function $r : [0,\infty) \to \Delta_I$ and a subsequence $\rho^{(n')}$ converging to r uniformly on compacts.

We show that r is a solution of the integral equation. We need to verify that for all $t \geq 0$

$$\lim_{n'\to\infty} \int_0^t \sum_{l=0}^{\infty} 1_{[\tau_{n',l},\tau_{n',l+1})}(u)\psi\left(\rho^{(n')}(\tau_{n',l}), \mathcal{T}_{n'}^u\right) du \quad = \quad \int_0^t \psi\left(r(u), T^u\right) du\,.$$

We obtain by the triangle inequality and Lipschitz continuity,

$$\begin{aligned}
&\left\|\int_0^t \sum_{l=0}^{\infty} 1_{[\tau_{n',l},\tau_{n',l+1})}(u)\, \psi\left(\rho^{(n')}(\tau_{n',l}), \mathcal{T}_{n'}^u\right) du \ - \ \int_0^t \psi\left(r(u), T^u\right) du\right\| \\
\leq\ & \int_0^t \left\|\sum_{l=0}^{\infty} 1_{[\tau_{n',l},\tau_{n',l+1})}(u)\, \psi\left(\rho^{(n')}(\tau_{n',l}), T^u\right) - \psi\left(r(u), T^u\right)\right\| du \\
&+ \int_0^t \sum_{l=0}^{\infty} 1_{[\tau_{n',l},\tau_{n',l+1})}(u) \left\|\psi\left(\rho^{(n')}(\tau_{n',l}), \mathcal{T}_{n'}^u\right) - \psi\left(\rho^{(n')}(\tau_{n',l}), T^u\right)\right\| du \\
\leq\ & Lt \cdot \max_{0\leq \tau_{n',l}\leq t} \ \sup_{\tau_{n',l}\leq u\leq \tau_{n',l+1}} \left\|\rho^{n'}(\tau_{n',l}) - r(u)\right\| + L\int_0^t \|\mathcal{T}_{n'}^u - T^u\|\, du.
\end{aligned}$$

r is uniformly continuous on compact sets. Thus, the first term converges to 0 by choice of $\rho^{(n')}$. The second term converges to 0, since the averages $\mathcal{T}_{n'}$ converge to T in $L^1([0,t])$ for any $t > 0$.[2] □

Proof of Theorem 3.3. The proof of the consistency of the Euler scheme can be divided into two steps. First, control the approximation error locally, and then find bounds on the global approximation error.

We choose the Lipschitz constant L and the bound B as in the proof of Proposition 3.2. For simplicity, we omit the index n from $t_{n,k}$. We define and bound a local approximation error $l_{n,k}$ as follows:

$$\begin{aligned}
l_{n,k} \quad &:= \quad \sup_{t_k\leq s\leq t_{k+1}} \left\|\int_{t_k}^s \psi(r(u), T^u)\, du - \int_{t_k}^s \psi(r(t_k), R^{(n),t_{k+1}})\, du\right\| \\
&\leq \quad L \cdot \left\{\int_{t_k}^{t_{k+1}} \|r(u) - r(t_k)\| du + \int_{t_k}^{t_{k+1}} \|T^u - R^{(n),t_{k+1}}\| du\right\} \qquad (39)
\end{aligned}$$

Here, we used the Lipschitz continuity of ψ. Now observe that (14) implies

$$\|r(u) - r(t_k)\| \leq \int_{t_k}^u \|\psi(r(v), T^v)\|\, dv \leq B(u - t_k).$$

[2]The L^1-convergence of the averages can be verified by Doob's martingale convergence theorem. See e.g. the proof of Proposition 3.7.

We get therefore for the right hand side of (39) an upper bound

$$\begin{aligned} & L \cdot \left\{ \int_{t_k}^{t_{k+1}} B(u - t_k)\, du + \int_{t_k}^{t_{k+1}} \|T^u - R^{(n),t_{k+1}}\| du \right\} \\ \leq \quad & L \cdot \left\{ \frac{B}{2n^2} + \int_{t_k}^{t_{k+1}} \|T^u - R^{(n),t_{k+1}}\| du \right\} \end{aligned} \tag{40}$$

Observe that for $t_k \leq s \leq t_{k+1}$ we can rewrite (13)

$$r^{(n)}(s) = r^{(n)}(t_k) + \int_{t_k}^{s} \psi(r^{(n)}(t_k), R^{(n),t_{k+1}})\, du \tag{41}$$

Next, we define the error

$$\delta_k = \delta_k^{(n)} = \|r(t_k) - r^{(n)}(t_k)\| \tag{42}$$

and observe that by (14) and (41) for every $t_k \leq s \leq t_{k+1}$

$$\begin{aligned} \|r(s) - r^{(n)}(s)\| \quad \leq \quad & \|r(t_k) - r^{(n)}(t_k)\| \\ & + \left\| \int_{t_k}^{s} \psi(r(u), T^u) - \psi(r^{(n)}(t_k), R^{(n),t_{k+1}})\, du \right\| \\ \leq \quad & \delta_k + \left\| \int_{t_k}^{s} \psi(r(t_k), R^{(n),t_{k+1}}) - \psi(r^{(n)}(t_k), R^{(n),t_{k+1}})\, du \right\| \\ & + \left\| \int_{t_k}^{s} \psi(r(u), T^u) - \psi(r(t_k), R^{(n),t_{k+1}})\, du \right\| \\ \leq \quad & \delta_k \left(1 + L\frac{1}{n}\right) + l_{n,k}. \end{aligned} \tag{43}$$

In particular, taking $s = t_{k+1}$ we get

$$\delta_{k+1} \leq \delta_k \left(1 + L\frac{1}{n}\right) + l_{n,k} \tag{44}$$

Observing $\delta_0 = 0$, we derive for $0 \leq k \leq \lfloor tn \rfloor + 1$ by induction

$$\delta_k \ \leq \ \left(1 + \frac{L}{n}\right)^k \sum_{m=0}^{k-1} l_{n,m} \tag{45}$$

Hence, we can estimate for $0 \leq k \leq \lfloor tn \rfloor + 1$

$$\begin{aligned} \delta_k \quad \leq \quad & \left(1 + \frac{L}{n}\right)^{\lfloor tn \rfloor + 1} \sum_{k=0}^{\lfloor tn \rfloor} l_{n,k} \\ \leq \quad & \left(1 + \frac{L}{n}\right)^{\lfloor tn \rfloor + 1} \cdot L \cdot \sum_{k=0}^{\lfloor tn \rfloor} \left(\frac{B}{2n^2} + \int_{t_k}^{t_{k+1}} \|T^u - R^{(n),t_{k+1}}\| du \right) \\ \leq \quad & D' \cdot \left(\frac{1}{n} + \int_0^{t+\frac{1}{n}} \|T^u - R^{(n),u}\| du \right) \end{aligned} \tag{46}$$

where D' is some constant depending only on t. This together with (43) implies that

$$\sup_{0 \leq s \leq t} \|r(s) - r^{(n)}(s)\| \leq D \cdot \left(\frac{1}{n} + \int_0^{t+\frac{1}{n}} \|T^u - R^{(n),u}\| du \right), \tag{47}$$

where D is some constant depending only on t. □

Proof of Corollary 3.4. Inequality (15) implies clearly the convergence of $r^{(n)}$ to r given appropriate conditions on the convergence of Y_t^n to 0 for any $t \in \mathbb{R}_+$. Since all quantities we are dealing with are uniformly bounded, convergence in any L^p-norm ($p \in [1, \infty)$) and convergence in probability are equivalent. $\square$

A.2 Proofs of Section 3.2

For technical reasons, we need the following concept of an *exhausting sequence.*

Definition A.1. *Let P be a probability measure and μ be a locally finite kernel from Ω to $\mathbb{R}_+$. A sequence $(C_N)_{N\in\mathbb{N}} \subseteq \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ is called* exhausting *for P and μ, if the following properties are satisfied.*

(1) $C_N \in \{F \times [0, \beta) : \; F \in \mathcal{F}, \; \beta > 0\}$ *for all N.*

(2) $P\mu(C_N) < \infty$.

(3) $\bigcup_N C_N = \Omega \times \mathbb{R}_+$.

Lemma A.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let μ be a locally finite transition kernel from Ω to $\mathbb{R}^+$. Then there exists an exhausting sequence $(C_N)_{N\in\mathbb{N}}$ for P and μ. Thus, Definition (20) defines a unique σ-finite measure $P\mu$ on the whole σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$.*

Proof. For $n, m \in \mathbb{N}$ define $B_{n,m} := \{\omega : \; \mu(\omega, [0, m)) \leq n\} \times [0, m)$. By definition

$$P\mu(B_{n,m}) = \int_{\mu(\omega,[0,m))\leq n} \mu(\omega, [0, m)) \, P(d\omega) \leq n \, .$$

Since $\mu(\omega, \cdot)$ is a locally finite measure on $\mathcal{B}(\mathbb{R}_+)$, we find

$$\bigcup_n B_{n,m} = \{\omega : \; \mu(\omega, [0, m)) < \infty\} \times [0, m) = \Omega \times [0, m) \, .$$

Thus, $\bigcup_{n,m} B_{n,m} = \Omega \times \mathbb{R}_+$.

Finally, Caratheodory's extension theorem implies that a σ-finite measure $P\mu$ on $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$ is uniquely specified by Definition 20. $\square$

Proof of Theorem 3.6. Let μ_k be the measure associated with the cumulative distribution function S_k. Define $\mu := \sum_{k=1}^{K} \mu_k$. By Assumption 3.5 $\mu_k(\omega, \cdot)$ is a locally finite measure on $\mathbb{R}_+$. The mapping $\mu_k(\cdot, B) : \Omega \mapsto \bar{\mathbb{R}}_+$ is measurable. The same is true for μ.

By Lemma A.2 both $P\mu$ and $P\mu_k$ are σ-finite measures. Moreover, $P\mu$ dominates $P\mu_k$. Thus, by the theorem of Radon-Nikodym, there exist densities $f_k : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(P\mu_k) = f_k d(P\mu)$ ($k = 1\ldots, K$).

For any $F \in \mathcal{F}$ we obtain

$$\int_F S_k^t(\omega) \, P(d\omega) = P\mu_k(F \times [0, t]) = \int_F \int 1_{[0,t]}(u) f(\omega, u) \, d\mu(\omega, u) \, P(d\omega) \, .$$

Since the equality holds for all $F \in \mathcal{F}$, we obtain (21).

Observe that $dP\mu = \sum_{k=1}^{K} dP\mu_k = \left(\sum_{k=1}^{K} f_k\right) dP\mu$. Hence, we can conclude that $\sum_{k=1}^{K} f_k = 1$ $P\mu$-almost everywhere.

Finally, let (C_N) be an exhausting sequence for P and μ. This implies that $\int 1_{C_N} f_k \, dP\mu = P\mu_k(C_N) \leq P\mu(C_N) < \infty$. Thus, the functions $1_{C_N} f_k$ are $P\mu$-integrable. $\square$

Proof of Proposition 3.7. By Lemma A.2 we can find an exhausting sequence $(C_N)_N$ for P and μ. It clearly suffices to verify the claim for $P\mu$-almost every $(\omega, s) \in C_N$ and any $N \in \mathbb{N}$. By definition, $C_N = F_N \times [0, \alpha_N)$ for some $F_N \in \mathcal{F}$ and $\alpha_N > 0$.

W.l.o.g. suppose that $P\mu(C_N) > 0$. Since $P\mu(C_N)$ is finite, we may normalize $P\mu$. Thus, we assume w.l.o.g. that $P\mu \in \mathcal{M}_1(C_N)$.

By $\mathcal{F}_n$ we denote the σ-algebra on $\mathbb{R}_+$ generated by the partition

$$\{\{0\}\} \cup \left\{ \left(\frac{l-1}{n}, \frac{l}{n} \right], \, l \in \mathbb{N} \right\}.$$

$\mathcal{F}_n$ induces a σ-algebra $\mathcal{G}_n^N$ on C_N, namely

$$\mathcal{G}_n^N = \Big(F_N \cap \mathcal{F} \Big) \otimes \Big([0, \alpha_N) \cap \mathcal{F}_n \Big),$$

where $F_N \cap \mathcal{F} = \{F_N \cap F : \, F \in \mathcal{F}\}$, $[0, \alpha_N) \cap \mathcal{F}_n = \{[0, \alpha_N) \cap E : \, E \in \mathcal{F}_n\}$, respectively.

Let $g : C_N \to \mathbb{R}$ be measurable and integrable with respect to $P\mu$. Doob's martingale convergence theorem for directed index sets implies that $E_{P\mu}(g|\mathcal{G}_n^N)$ converges $P\mu$-almost surely to g as $n \to \infty$. This result can be applied to $1_{C_N} f_k$, since $E_{P\mu}(1_{C_N} f_k) < \infty$ by Theorem 3.6.

For $(\omega, s) \in C_N$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the number $(\lfloor sn \rfloor + 1)/n$ is strictly smaller than α_N. We obtain therefore for $n \geq n_0$

$$\begin{aligned}
R_k^{(n)}(\omega, s) &= \frac{S_k^{(\lfloor sn \rfloor)+1/n} - S_k^{\lfloor sn \rfloor/n}}{\sum_{l=1}^K \left(S_l^{(\lfloor sn \rfloor+1)/n} - S_l^{\lfloor sn \rfloor/n} \right)} \\
&= \left(\int \mathbf{1}_{(\lfloor sn \rfloor/n, (\lfloor sn \rfloor+1)/n]} f_k d\mu \right) \cdot \left(\sum_{l=1}^K \int \mathbf{1}_{(\lfloor sn \rfloor/n, (\lfloor sn \rfloor+1)/n]} f_l d\mu \right)^{-1} \\
&= \big(E_{P\mu}(f_k|\mathcal{G}_n^N)(\omega, s) \big) \cdot \left(\sum_{l=1}^K E_{P\mu}(f_l|\mathcal{G}_n^N)(\omega, s) \right)^{-1}
\end{aligned}$$

The last term converges $P\mu$-almost everywhere to $f_k(\omega, s) \cdot \left(\sum_{l=1}^K f_l(\omega, s) \right)^{-1}$. □

Proof of Corollary 3.8. By Proposition 3.7 we obtain that $\lim_{n\to\infty} R^{(n)}(\omega, s) = T(\omega, s)$ $P\mu$-almost everywhere. Set $L \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ be the set of all (ω, s) such that $\lim_{n\to\infty} R^{(n)}(\omega, s)$ exists and equals $T(\omega, s)$. Denote by L^c the complement of L, and let $B_\omega = \{s : \, (\omega, s) \notin L\}$. Then,

$$0 = \int L^c \, dP\mu = \int \mu(\omega, B_\omega) \, P(dw).$$

Hence, $\mu(\omega, B_\omega) = 0$ for P-almost all $\omega \in \Omega$. Since $\mu(\omega, \cdot)$ dominates the Lebesgue measure for P-almost all $\omega \in \Omega$, we obtain that $\lambda(B_\omega) = 0$ for P-almost all $\omega \in \Omega$ where λ is the Lebesgue measure. Thus, we conclude that P-almost surely for $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty} Y_t^n = \lim_{n\to\infty} \int \|T^u - R^{(n),u}\| du = \int \lim_{n\to\infty} \|T^u - R^{(n),u}\| du = 0.$$

Interchanging limit and integral is justified by the dominated convergence theorem, since P-almost surely T and $R^{(n)}$ $(n \in \mathbb{N})$ are bounded in Δ_K. The result follows from Theorem 3.3. □

Proof of Proposition 3.10. Observe that

$$0 = P\mu\left(\sum_{k=1}^{K} f_k(\omega, u) = 0\right) = \int \mu\left(\omega, \left\{s : \sum_{k=1}^{K} f_k(\omega, s) = 0\right\}\right) P(d\omega).$$

Thus, the sum of the functions $(f_k(\omega))_{k=1,...,K}$ is $\mu(\omega, \cdot)$-almost surely positive for P-almost all $\omega \in \Omega$.

Assumption 3.5 implies that the complement of any $\mu(\omega, \cdot)$-nullset lies densely in $\mathbb{R}_+$. The regularity of f implies then the positivity of the sum of the components Lebesgue-almost everywhere with probability one.

Let $B_n(s) := \left[\frac{\lfloor sn \rfloor - 1}{n}, \frac{\lfloor sn \rfloor + 1}{n}\right]$. Note that $\mu(t, t+\epsilon) > 0$ for $t \in \mathbb{R}_+$ and $\epsilon > 0$. Thus, by definition of $R^{(n)}$ it holds that

$$\frac{\inf_{u \in B_n(s)} f_k(u)}{\sum_{l=1}^{K} \sup_{u \in B_n(s)} f_l(u)} \leq R_k^{(n)}(s) \leq \frac{\sup_{u \in B_n(s)} f_k(u)}{\sum_{l=1}^{K} \inf_{u \in B_n(s)} f_l(u)}.$$

Since f is Lebesgue-almost everywhere continuous, the claim follows. □

Proof of Corollary 3.11. The proof is analogous to the last part of the proof of Corollary 3.8. □

A.3 Proofs of Section 4.2

Proof of Lemma 4.5. The equation $N = -\nabla\Phi$ is easily verified. In particular, we obtain on the simplex Δ_I the inequality

$$\dot{\Phi}(r) = \nabla_r \Phi(r)\psi(r) = -\sum_{i=1}^{I} r_i N_i^2(r) \leq 0.$$

The convexity of Φ follows from the concavity of the logarithm. □

Proof of Corollary 4.6. We denote the minimal attractor of Δ_I by $\tilde{\mathcal{B}}$. The inclusion $\tilde{\mathcal{B}} \subseteq \mathcal{A}$ is implied by LaSalle's Corollary 4.4, since $\dot{\Phi}(r) = -\sum r_i N_i^2(r)$. Conversely, the condition $\sum r_i N_i^2(r) = 0$ implies

$$\forall i = 1, 2, \ldots, I: \quad r_i = 0 \;\vee\; N_i(r) = 0.$$

Thus, $\psi(r) = 0$ for $r \in \mathcal{A}$. $\mathcal{A}$ is therefore a set of fix points for the flow ϕ, hence $\mathcal{A} \subseteq \tilde{\mathcal{B}}$. □

A.4 Results related to the global attractor

Proposition A.3. *Suppose that Assumptions 2.1 and 4.1 are satisfied. Then*

$$\overline{ri\,(\Delta_I) \cap \mathcal{A}} \subseteq \mathcal{B} \subseteq \mathcal{A}.$$

Moreover, $\mathcal{A}_{\min} \subseteq \mathcal{A}$ is a non empty, closed, convex set of fixed points for ϕ, and the following holds:

(1) $\overline{ri\,(\Delta_I) \cap \mathcal{A}} \subseteq \mathcal{A}_{\min}$.

(2) The converse inclusion $\mathcal{A}_{\min} \subseteq \overline{ri\,(\Delta_I) \cap \mathcal{A}}$ holds, if and only if the set $\mathcal{A}_{\min} \cap ri\,(\Delta_I)$ is non empty. In this case, $\mathcal{A}_{\min} = \overline{ri\,(\Delta_I) \cap \mathcal{A}} = \overline{ri\,(\Delta_I) \cap \mathcal{A}_{\min}}$.

(3) Moreover, if $\mathcal{A}_{\min} \cap ri\,(\Delta_I)$ is non empty, then $\mathcal{A}_{\min} \subseteq \mathcal{B}$.

Proof. Because $\mathcal{A}$ is a set of fixed points, $\text{ri}\,(\Delta_I) \cap \mathcal{A} \subseteq \mathcal{B}$. As $\mathcal{B}$ is closed, the first inclusion is proved. The second inclusion $\mathcal{B} \subseteq \mathcal{A}$ is a consequence of Corollary 4.6, since $\mathcal{A}$ is closed.

Since the Lyapunov function Φ is continuous on Δ_I, $\mathcal{A}_{\min}$ is closed and non empty. The convexity of Φ implies that $\mathcal{A}_{\min}$ is convex.

We claim that $\mathcal{A}_{\min} \subseteq \mathcal{A}$. Otherwise, by continuity of N there exist an initial value $r_0 \in \mathcal{A}_{\min}$ with a neighborhood $\mathcal{N}(r_0)$ and $\delta > 0$ such that $\sum_{i=1}^{I} r_i N_i(r)^2 \;>\; \delta >\; 0$ for all $r \in \mathcal{N}(r_0) \cap \Delta_I$. As the flow is continuous, we can find $\epsilon > 0$ such that $\phi_t(r_0) \in \mathcal{N}(r_0)$ for all $0 \leq t \leq \epsilon$. Therefore

$$
\begin{aligned}
\Phi(\phi_\epsilon(r_0)) &= \Phi(r_0) + \int_0^\epsilon \dot{\Phi}(\phi_s(r_0))\, ds \\
&= \Phi(r_0) - \int_0^\epsilon \sum_{i=1}^{I} \phi_{i,s}(r_0) N_i(\phi_s(r_0))^2\, ds \\
&\leq \Phi(r_0) - \delta\,\epsilon \;< \Phi(r_0),
\end{aligned}
$$

a contradiction. It follows that $\mathcal{A}_{\min} \subseteq \mathcal{A}$. Hence, $\mathcal{A}_{\min}$ is a set of fixed points.
Ad (1). W.l.o.g. suppose $\text{ri}\,(\Delta_I) \cap \mathcal{A} \neq \emptyset$. If $r \in \text{ri}\,(\Delta_I) \cap \mathcal{A}$, then $N_i(r) = 0$ for all $i \in I$ by Corollary 4.6. Thus, for all $s \in \Delta_I$

$$
\Phi(s) = \Phi(s) + \sum_{i=1}^{I} (s_i - r_i) N_i(r) = \Phi(s) - \nabla\Phi(r)(s - r) \geq \Phi(r)
$$

by the subgradient inequality for convex functions. Hence $r \in \mathcal{A}_{\min}$. Since $\mathcal{A}_{\min}$ is closed, the claim follows.
Ad (2). If $\mathcal{A}_{\min} \subseteq \overline{\mathcal{A} \cap \text{ri}\,(\Delta_I)}$, then $\mathcal{A} \cap \text{ri}\,(\Delta_I) \neq \emptyset$. However, the points in $\mathcal{A} \cap \text{ri}\,(\Delta_I)$ are minima. Hence $\mathcal{A}_{\min} \cap \text{ri}\,(\Delta_I) \neq \emptyset$.

On the other hand, if $\mathcal{A}_{\min} \cap \text{ri}\,(\Delta_I) \neq \emptyset$, let $r \in \mathcal{A}_{\min} \cap \text{ri}\,(\Delta_I)$. If $s \in \mathcal{A}_{\min}$, then $\alpha s + (1-\alpha) r \in \mathcal{A}_{\min} \cap ri(\Delta_I)$ for $\alpha \in (0, 1]$. Hence, $\mathcal{A}_{\min} = \overline{\mathcal{A}_{\min} \cap \text{ri}\,(\Delta_I)}$. Finally, observe that $\mathcal{A}_{\min} \subseteq \mathcal{A}$.
Ad (3). The claim is immediate from $\overline{\text{ri}\,(\Delta_I) \cap \mathcal{A}} \subseteq \mathcal{B}$ and part (2). □

Proposition A.4. *Any $r \in \mathcal{A} \cap \partial^*(\Delta_I)$ is contained in the relative interior $ri\,(\Delta_J)$ for some $J \subseteq I$, $J \neq I$. r minimizes the Lyapunov function Φ on Δ_J.*

Proof. r is clearly contained in the relative interior of some subsimplex.

Moreover, we have that $r = \sum_{i \in J} r_i e_i$, $r_i > 0$ $(i \in J)$. Since $r \in \mathcal{A}$, we obtain $N_i(c) = 0$ for all $i \in J$. Hence, for all $x = \sum_{i \in J} x_i e_i \in \Delta_J$

$$
\Phi(x) = \Phi(x) + \sum_{j \in J} (x_i - r_i) N_i(r) = \Phi(x) - \nabla\Phi(r)(x - r) \geq \Phi(r)
$$

by the subgradient inequality for convex functions. □

We need the following technical lemma.

Lemma A.5. *Let Assumptions 2.1 and 4.1 be satisfied and assume that $\mathcal{B} \setminus \mathcal{A}_{\min} \neq \emptyset$. There exist $g \in \mathcal{B} \setminus \mathcal{A}_{\min}$ and $r \in ri(\Delta_I)$ with $g \in \omega(r)$. $\mathcal{C} := \omega(r) \subseteq \mathcal{B} \backslash \mathcal{A}_{\min} \cap \partial^*(\Delta_I)$ is a non empty, connected set satisfying the following properties:*

(C_1) $\forall c \in \mathcal{C}\; \forall J \subseteq I \left(c \in ri\,(\Delta_J) \;\Rightarrow\; \Phi(c) = \min_{d \in \Delta_J}\; \Phi(d) \right)$.

(C_2) $\forall c \in \mathcal{C}\ \exists i \in I\ N_i(c) > 0$.

(C_3) $\forall c \in \mathcal{C}\ \forall i \in I\ \left(N_i(c) > 0\ \Rightarrow\ \exists d \in \mathcal{C}\ N_i(d) = 0\right)$.

Proof. Let $\mathcal{B} \backslash \mathcal{A}_{\min} \neq \emptyset$. Then there exist $g \in \mathcal{B} \backslash \mathcal{A}_{\min}$ and $r \in \text{ri}\ (\Delta_I)$ such that $g \in \omega(r)$. Suppose not: Then $\Phi(g) = \min_{x \in \Delta_I} \Phi(x)$ for all $r \in \text{ri}\ (\Delta_I)$ and for all $g \in \omega(r)$. Because Φ is continuous, $\Phi(s) = \min_{x \in \Delta_I} \Phi(x)$ for all $s \in \mathcal{B}$. Thus, $\mathcal{B} \backslash \mathcal{A}_{\min} = \emptyset$, a contradiction.

Define $\mathcal{C} := \omega(r)$. Since Δ_I is compact and invariant, $\mathcal{C}$ is a non empty connected set contained in Δ_I ([(Amann 1983)], Theorem (17.2)). Moreover, $\mathcal{C} \subseteq \mathcal{B} \backslash \mathcal{A}_{\min}$, since Φ is constant on $\omega(r)$. The inclusion $\mathcal{C} \subseteq \partial^*(\Delta_I)$ follows from $\mathcal{C} \subseteq \mathcal{A}$ and property (C_2) proven below.

Ad (C_1). Property (1) is a direct consequence of Proposition A.4.

Ad (C_2). Suppose that there exists $c = \sum_{i \in I} c_i e_i \in \mathcal{C}$ such that $N_i(c) \leq 0$ for all $i \in I$. Let $x = \sum_{i \in J} x_i e_i \in \Delta_I$. Because $c \in \mathcal{A}$, we obtain

$$\sum_{i \in I} (x_i - c_i)\ N_i(c) = \sum_{i \in I} x_i\ N_i(c) \leq 0.$$

Hence, with the same argument as in (1)

$$\Phi(x) \geq \Phi(x) + \sum_{i \in I} (x_i - c_i) N_i(c) = \Phi(x) - \nabla\Phi(c)(x - c) \geq \Phi(c).$$

Thus, $c \in \mathcal{A}_{\min}$, a contradiction.

Ad (C_3). Let $c \in \mathcal{C}$ and $N_i(c) > 0$. Since $c \in \mathcal{A}$, it follows that $c_i = 0$. Since $c \in \omega(r)$, we find a strictly increasing sequence $(t_{2k}) \subseteq \mathbb{R}_+$ such that $\lim_{k \to \infty} t_{2k} = \infty$ and $\lim_{k \to \infty} \phi_{t_{2k}}(r) = c$. Since $0 = c_i = \lim_{k \to \infty} \phi_{i, t_{2k}}(r)$, we may assume that for all $k \in \mathbb{N}_0$

$$\phi_{i, t_{2(k+1)}}(r) \leq \phi_{i, t_{2k}}(r). \tag{48}$$

Recall that D is the open extended state space of the flow as defined in (27). Let

$$G = \{g \in D : \ N_i(g) > 0\}.$$

Then G is an open neighborhood of c, and therefore $\phi_{t_{2k}}(r) \in G$ for k sufficiently large. W.l.o.g. assume that $\phi_{t_{2k}}(r) \in G$ for all k. Define the exit time from the set G by

$$t_{2k+1} = \inf\{t \geq t_{2k} : \ N_i(\phi_t(r)) \leq 0.\}$$

Note that for all $s \in [t_{2k}, t_{2k+1})$

$$\dot{\phi}_{i,s}(r) = \phi_{i,s}(r)\ N_i(\phi_s(r)) > 0.$$

Therefore, $[t_{2k}, t_{2k+1}) \ni s \mapsto \phi_{i,s}(r)$ is strictly increasing. Since (48) holds, we obtain that t_{2k+1} must be strictly smaller that $t_{2(k+1)}$. Thus, $t_{2k} < t_{2k+1} < t_{2k+2}$. By continuity of N we obtain that $N_i(\phi_{t_{2k+1}}(r)) = 0$ for all k. Because $\{g \in \Delta_I : N_i(g) = 0\} := \mathcal{C}'$ is compact, there exists an element $d \in \mathcal{C}'$ such that it is the limit of an appropriate subsequence of $\phi_{t_{2k+1}}(r)$, namely $d = \lim_{k' \to \infty} \phi_{t_{2k'+1}}(r)$, where k' is some sequence of natural numbers converging to infinity. We obtain that $d \in \mathcal{C} = \omega(r)$ with $N_i(d) = 0$. Hence, property (C_3) is proven. $\square$

A.5 Proofs of Section 4.3

Proof of Theorem 4.8. The theorem is a consequence of Lemma A.5 which is proven in Section A.4. First assume that (1) holds. Suppose that $\mathcal{B}\backslash\mathcal{A}_{\min} \neq \emptyset$. By Lemma A.5 there exists a connected set $\mathcal{C} \subseteq \partial^*(\Delta_I)$ satisfying the properties (C_1) and (C_2) and (C_3). The properties (C_2) and (C_3) imply that $\mathcal{C}$ contains at least two points.

Define $M_n = \bigcup_{\substack{J\subseteq I\\ |J|\leq n}} \Delta_J$. Note that $\mathcal{C} \subseteq \partial^*(\Delta_I) = M_{I-1}$. Take the minimal n such that $\mathcal{C} \subseteq M_n$. By minimality of n we find $J \subseteq I$ such that $|J| = n$ and $\mathcal{C} \cap \mathrm{ri}\ (\Delta_J) \neq \emptyset$. Because all points $\mathcal{C} \cap \mathrm{ri}\ (\Delta_J)$ are minima of Φ on Δ_J and Φ is strictly convex on Δ_J, we obtain $|\mathcal{C} \cap \mathrm{ri}\ (\Delta_J)| = 1$, a contradiction, since $|\mathcal{C}| \geq 2$ and $\mathcal{C}$ connected.

Now assume that (2) holds. Again by the subgradient inequality we obtain for all $c \in \mathcal{A} \setminus \mathcal{A}_{\min}$ and for all $i \in I$,

$$N_i(c) = N_i(c) - \sum_{j=1}^{I} c_j N_j(c) = \nabla\Phi(c)(e_i - c) \geq \Phi(c) - \Phi(e_i) > 0.$$

Hence, a set $\mathcal{C}$ as stated in Lemma A.5 satisfying the properties (C_2) and (C_3) simultanously cannot exist.

If additionally $\mathcal{A}_{\min}$ contains points of the relative interior of Δ_I, then the equality $\mathcal{B} = \mathcal{A}_{\min}$ is implied by Proposition A.3(3). □

Proof of Proposition 4.9. W.l.o.g. assume that $i = I$, and set $M := M^{(I)}$.
Ad (1). By (31), $\Phi(r) = \tilde{\Phi}(L(r))$ with

$$L(r) = M \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_{I-1} \end{pmatrix} + \lambda_I. \tag{49}$$

Let $r, r' \in \Delta_I$, $r \neq r'$, $\alpha \in (0,1)$. If $d = 0$, L is injective on Δ_I, thus

$$\begin{aligned} \Phi(\alpha r + (1-\alpha)r') \ &= \ \tilde{\Phi}(\alpha L(r) + (1-\alpha)L(r')) \\ > \ \alpha\tilde{\Phi}(L(r)) + (1-\alpha)\tilde{\Phi}(L(r')) \ &= \ \alpha\Phi(r) + (1-\alpha)\Phi(r'). \end{aligned}$$

Thus, Φ is strictly convex.
Ad (2). The mapping $L' : \mathbb{R}^I \to \mathbb{R}^K,\ r \mapsto L(r) - \lambda_I$ is linear with $\dim(\ker L') \geq 3$. Let $J = I \setminus \{I\}$, $r \in ri(\Delta_J)$, $N_J = \mathrm{span}\{(e_j - r) :\ j \in I \setminus \{I\}\}$. Then $\dim N_J = I - 2$. Thus,

$$\dim(N_J \cap \ker L') \geq \dim N_J + \dim(\ker L') - I = 1.$$

Let $v \in (N_J \cap \ker L') \setminus \{0\}$. Then for $\delta > 0$ sufficiently small, $\epsilon \in [0, \delta]$, we obtain that $r + \epsilon v \in \Delta_J$. Moreover, $L(r + \epsilon v) = L(r)$. This implies $\Phi(r + \epsilon v) = \Phi(v)$. Hence, Φ is not strictly convex on Δ_J.
Ad (3). First, let Φ be strictly convex on the boundary. Suppose there exists $v \in G$ such that the subspace

$$Q_v = \ker M \cap \mathrm{span}\{u - \bar{u} :\ u, \bar{u} \in G \setminus \{v\}\}$$

is non trivial. Then, we can find $w \in Q_v \setminus \{0\}$.

If $v \neq 0$, then $v = e_j$ for some $j \in I$. Otherwise, let $j = I$. Let $J := I \setminus \{j\}$, $r \in ri(\Delta_J)$. Define

$$\bar{w} = \begin{pmatrix} w \\ -\sum_{k=1}^{I-1} w_k \end{pmatrix}.$$

It is easy to see that $r + \epsilon\bar{w} \in \Delta_J$, if $|\epsilon|$ is sufficiently small, say $|\epsilon| < \delta$ for some $\delta > 0$. We obtain that $L(r + \epsilon\bar{w}) = L(r)$. This implies that $\Phi(r + \epsilon\bar{w}) = \Phi(r)$. Hence, Φ is not strictly convex on Δ_J.

The converse can be verified as follows. We set $J = I \setminus \{e_j\}$, $j \in I$. Let $r, r' \in \Delta_J$. Suppose that $L(r) = L(r')$. Then

$$w := \begin{pmatrix} r_1 - r'_1 \\ \vdots \\ r_{I-1} - r'_{I-1} \end{pmatrix} \in \ker M.$$

Let $v = e_j$, if $j \neq I$; otherwise, $v = 0$. Then $w \in \mathrm{span}\{u - \bar{u} : \; u, \bar{u} \in G \setminus \{v\}\}$, hence $w = 0$. This implies that L is injective on Δ_J. Analogous to part (1), it follows that Φ is strictly convex on Δ_J. $\square$

A.6 Results beyond Section 4.3

The analysis of the attractor $\mathcal{B}$ of the relative interior of the simplex Δ_I is more complicated, if $\mathcal{B} \not\subseteq \mathcal{A}_{\min}$. In this case, observe that the long-run wealth distributions $\mathcal{B}$ can be decomposed as

$$\mathcal{B} \;=\; (\mathcal{B} \cap \mathcal{A}_{\min}) \;\cup\; (\mathcal{B} \setminus \mathcal{A}_{\min}) .$$

Apart from the minima of the Lyapunov function, we thus need to investigate the set $\mathcal{B} \setminus \mathcal{A}_{\min}$.

Proposition A.6. *$\mathcal{B} \setminus \mathcal{A}_{\min}$ is a subset of the boundary $\partial^*(\Delta_I)$. Any $r^* \in \mathcal{B} \setminus \mathcal{A}_{\min}$ is contained in the relative interior ri (Δ_J) of some subsimplex for some $J \subseteq I$, $J \neq I$. r^* minimizes the Lyapunov function Φ on ri (Δ_J).*

Proof. Since $\mathcal{B} \subseteq \mathcal{A}$, we get that $\mathcal{B} \setminus \mathcal{A}_{\min} \subseteq \mathcal{A} \setminus \mathcal{A}_{\min}$. Since ri $(\Delta_I) \cap \mathcal{A} \subseteq \mathcal{A}_{\min}$ by Proposition A.3(1), we obtain that $\mathcal{B} \setminus \mathcal{A}_{\min} \subseteq \partial^*(\Delta_I)$. Since $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B} \setminus \mathcal{A}_{\min} \subseteq \mathcal{A} \cap \partial^*(\Delta_I)$. Thus, the properties of r^* are implied by Proposition A.4. $\square$

According to the last proposition the elements of $\mathcal{B} \setminus \mathcal{A}_{\min}$ are included in the relative interior of some subsimplex. The following proposition further investigates the boundary of Δ_I and provides conditions when boundary points cannot be included in $\mathcal{B} \setminus \mathcal{A}_{\min}$. To simplify the notation, we assume that $r^* \in$ ri (Δ_J) for some J which includes index I. Otherwise, we can relabel the unit vectors. We set $M := M^{(I)}$ as defined in (32).

Proposition A.7. *Suppose that*

$$\mathit{span}\;\left((e_j)_{j \in J \setminus \{I\}}, \ker M\right) = \mathbb{R}^{I-1}. \tag{50}$$

Then the following holds.

(1) r^* is not included in $\mathcal{B} \setminus \mathcal{A}_{\min}$.

(2) If r^* is a minimizer of the Lyapunov function on ri (Δ_J), then $r^* \in \mathcal{A}_{\min}$.

Proof. We prove the second part first. The first part is then an elementary consequence.
Ad (2). $\Phi(r) = \tilde{\Phi}(L(r))$ with L defined according to equation (49). We can find $v \in \ker M$ such that

$$\sum_{j \in J \setminus \{I\}} r^*_j \cdot e_j + v \in ri(conv\{0, e_1, \ldots, e_{I-1}\})).$$

Let $u = r^* + \begin{pmatrix} v \\ -\sum_{j=1}^{I-1} v_j \end{pmatrix} \in ri(\Delta_I)$. Then $\Phi(u) = \Phi(r^*)$.

Let $w \in \mathbb{R}^I$ such that $\sum_{j=1}^{I} w_j = 0$. Then there exist $w^1 \in \text{span}\{(e_j)_{j \in J \setminus \{I\}}\} \subseteq \mathbb{R}^{I-1}$, $w^2 \in \ker M \subseteq \mathbb{R}^{I-1}$ such that

$$\begin{pmatrix} w_1 \\ \vdots \\ w_{I-1} \end{pmatrix} = w^1 + w^2.$$

For $|\epsilon|$ sufficiently small, we obtain that $u + \epsilon \begin{pmatrix} w \\ -\sum_{j=1}^{I-1} w_j \end{pmatrix} \in \text{ri}\ (\Delta_I)$ and that

$$\Phi(u + \epsilon w) = \tilde{\Phi}\left(L \left(\sum_{j=1}^{I-1} (r^* + v + \epsilon w^1 + \epsilon w^2)_j \cdot e_j \right) \right)$$

$$= \quad \Phi\left(r^* + \epsilon \begin{pmatrix} w^1 \\ -\sum_{j=1}^{I-1} w_j^1 \end{pmatrix} \right) \ \geq \ \Phi(r^*) \ = \ \Phi(u).$$

The inequality follows from the fact that $r^* + \begin{pmatrix} w^1 \\ -\sum_{j=1}^{I-1} w_j^1 \end{pmatrix} \in \Delta_J$.

We obtain that u is a local minum of Φ on Δ_I. Hence, u is a global minimum. From this follows that $r^* \in \text{argmin}_{r \in \Delta_I} \Phi(r)$.

Ad (1). Suppose $r^* \in \mathcal{B} \setminus \mathcal{A}_{\min}$. Then $r^* \in \text{argmin}_{r \in ri(\Delta_J)} \Phi(r)$ for some $J \subseteq I$, $J \neq I$ by Proposition A.6. Then part (2) implies that $r^* \in \mathcal{A}$, a contradiction. ☐

Remark A.8. *Proposition A.7 provides essentially a condition in terms of the dimension of the kernel of the matrix M. If $\dim(\ker M)$ is large enough, then (50) is generically satisfied. This parallels the results of Proposition 4.9.*

A.7 Proofs of Section 4.4

Proof of Lemma 4.10. Part (1) is implied by the strict convexity of $\tilde{\Phi}$ and the first order conditions. We only need to show that (2) holds. The set $\Lambda \subseteq \Delta_K$ is a compact set included in ri (Δ_K). Since $\tilde{\Phi}$ restricted to Λ is continuous, there exists a global minimum attained at some point $x^* \in \Lambda$. Assumption (4.1) ensures that $\tilde{\Phi}$ is strictly convex. Moreover, Λ is convex. This implies the uniqueness of x^*. Finally, if $\pi \in \Lambda$, then clearly $x^* = \pi$. Conversely, if $\pi \notin \Lambda$, then $\pi \neq x^* \in \Lambda$. ☐

Proof of Lemma 4.11. $\mathcal{A}_{\min}$ is non empty by Proposition A.3. Representation (34) is an immeditate consequence of (31) and (33), as the following calculation shows:

$$\min_{r \in \Delta_I} \Phi(r) = \min_{r \in \Delta_I} \tilde{\Phi}\left(\left(\sum_{j=1}^{I} r_j \lambda_{j,k} \right)_{k=1,\dots,K} \right) = \min_{x \in \Lambda} \tilde{\Phi}(x) = \tilde{\Phi}(x^*). \tag{51}$$

The solution of the linear system in $\mathbb{R}^I$, given by $\sum_{i=1}^{I} r_i \lambda_i = x^*$ with r unknown, is an affine subspace of $\mathbb{R}^I$. This implies that $\mathcal{A}_{\min}$ is the intersection of a simplex and an affine subspace, hence polyhedral. ☐

Proof of Proposition 4.12.
(i) ⇒ (ii): Suppose not. Then there exists $j \in I \setminus \{i\}$ such that

$$\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i)<0.$$

Define for $\alpha\in[0,1]$ the vector $x(\alpha):=\alpha\lambda_j+(1-\alpha)\lambda_i\in\Lambda$. Then

$$\frac{d}{d\alpha}\tilde{\Phi}(x(\alpha))_{|\alpha=0}=\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i)<0.$$

Since $\alpha\mapsto\frac{d}{d\alpha}\tilde{\phi}(x(\alpha))$ is continuous, we can find $1\geq\epsilon>0$ such that $\alpha\mapsto\tilde{\Phi}(x(\alpha))$ is strictly decreasing on $[0,\epsilon]$. Thus, $\tilde{\Phi}(x(\epsilon))<\tilde{\Phi}(\lambda_i)$. This implies $\lambda_i\neq x^*$, a contradiction.
(ii) ⇒ (i): Let $x=\sum_{j=1}^I r_j\lambda_j$ with $r_j\geq 0$ $(j\in I)$, $\sum_{j=1}^I r_j=1$. Then

$$\nabla\tilde{\Phi}(\lambda_i)\cdot(x-\lambda_i)=\sum_{j=1}^I r_j\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i)\geq 0.$$

Thus, by the subgradient inequality for convex functions

$$\tilde{\Phi}(x)\geq\tilde{\Phi}(\lambda_i)+\nabla\tilde{\Phi}(\lambda_i)\cdot(x-\lambda_i)\geq\tilde{\Phi}(\lambda_i).$$

Since the minimum of $\tilde{\Phi}$ is unique, we obtain $\lambda_i=x^*$.
(ii) ⇔ (iii):

$$\begin{aligned}\nabla\tilde{\Phi}(\lambda_i)\cdot(\lambda_j-\lambda_i) &= \sum_{k=1}^K\left(-\frac{\pi_k}{\lambda_{i,k}}+1\right)\cdot(\lambda_{j,k}-\lambda_{i,k})\\ = -\sum_{k=1}^K\pi_k\cdot\frac{\lambda_{j,k}}{\lambda_{i,k}}+\sum_{k=1}^K\pi_k &= 1-\sum_{k=1}^K\pi_k\frac{\lambda_{j,k}}{\lambda_{i,k}}.\end{aligned}$$

This clearly implies the equivalence of (ii) and (iii). ☐

Proof of Corollary 4.13. $\mathcal{A}_{\min}=\left\{r\in\Delta_I:\ \sum_{j=1}^I r_j\lambda_j=\lambda_i\right\}$ by (34). Then $e_i\in\mathcal{A}_{\min}$, hence $\mathcal{B}\subseteq\mathcal{A}_{\min}$ by Theorem 4.8(2). The last claim follows from Proposition A.3. ☐

Proof. 4.14 If λ_i is an extremal point of the polyhedron Λ, then $\mathcal{A}_{\min}=\{e_i\}$. Since $\emptyset\neq\mathcal{B}\subseteq\mathcal{A}_{\min}$, we obtain $\mathcal{B}=\mathcal{A}_{\min}$. ☐

Proof of Proposition 4.16. Since $\nabla\tilde{\Phi}(x^*)\cdot(\lambda_i-x^*)=1-\sum_{k=1}^K\pi_k\cdot\frac{\lambda_{i,k}}{x_k^*}$, we obtain

$$\nabla\tilde{\Phi}(x^*)\cdot(\lambda_i-x^*)\neq 0.$$

Let now $y\in\Lambda$. Assume that

$$\nabla\tilde{\Phi}(x^*)\cdot(y-x^*)<0.$$

For $\alpha\in[0,1]$ define the vector $x(\alpha):=\alpha y+(1-\alpha)x^*\in\Lambda$. The same arguments as in the part (i)⇒(ii) of the proof of Proposition 4.12 show that there exists $0<\epsilon\leq 1$ such that $\tilde{\Phi}(x(\epsilon))<\tilde{\Phi}(x^*)$. This implies that $x^*\neq\operatorname{argmin}_{x\in\Lambda}\tilde{\Phi}(x)$, a contradiction. Hence, for $y\in\Lambda$,

$$\nabla\tilde{\Phi}(x^*)\cdot(y-x^*)\ \geq\ 0, \tag{52}$$

$$\nabla\tilde{\Phi}(x^*)\cdot(\lambda_i-x^*)\ >\ 0. \tag{53}$$

Now, let $r \in \Delta_I$ such that $x^* = \sum_{j=1}^{I} r_j \lambda_j$. Then,

$$0 = \nabla\tilde{\Phi}(x^*) \cdot (x^* - x^*) = \sum_{j=1}^{I} r_j \nabla\tilde{\Phi}(x^*) \cdot (\lambda_j - x^*).$$

Since each summand is non negative by (52), we obtain that

$$r_i \nabla\tilde{\Phi}(x^*) \cdot (\lambda_i - x^*) = 0.$$

Finally, (53) implies that $r_i = 0$. □

A.8 Proofs of Section 4.5

Proof of Proposition 4.17. Clearly, $q_k \neq 0$ for all $k = 1, 2, \dots, K$. Namely, if $q_k = 0$, the demand for asset k is strictly positive (even infinite). Thus, $q_k \neq 0$ by (37), a contradiction. Assume that $\frac{\pi_k}{q_k} > \frac{\pi_l}{q_l}$ for some $l, k = 1, 2, \dots, K$. Then clearly $\lambda_{j,l} = 0$ for $j = 1, 2, \dots, I$, since agents are maximizers. Thus, $q_k = 0$ by (37), a contradiction. Hence, $\frac{\pi_k}{q_k} = \frac{\pi_l}{q_l}$ $(l, k = 1, 2, \dots, K)$. Since $\sum_{k=1}^{K} \pi_k = \sum_{k=1}^{K} q_k$, this implies that $\pi_k = q_k$ $(k = 1, 2, \dots, K)$. By (37) we obtain that

$$\mathcal{E} \subseteq \left\{ (\lambda_1, \lambda_2, \dots, \lambda_I) \in (\Delta_K)^I : \sum_{i=1}^{I} \lambda_{i,k} r_i = \pi_k \text{ for all } 1 \leq k \leq K \right\} =: \mathcal{E}'.$$

Clearly, for $(\lambda_1, \lambda_2, \dots, \lambda_I) \in \mathcal{E}'$ the price vector q equals π. For $q = \pi$ agents are indifferent between all strategies, thus no profitable deviation exists for any agent. Hence, $\mathcal{E}' \subseteq \mathcal{E}$.

For $(\lambda_1, \lambda_2, \dots, \lambda_I) \in \mathcal{E}$ we obtain $q = \pi$, thus $V_i^q(\lambda_i) = 0$ for all $i = 1, 2, \dots, I$. This implies that the wealth vector $(r_i)_{i=1,2,\dots,I}$ of the investors is constant. □

References

Algoet, Paul H. & Thomas M. Cover (1988), 'Asymptotic optimality and asymptotic equilibrium', *Annals of Probability* **16**, 876–898.

Amann, Herbert (1983), *Gewöhnliche Differentialgleichungen*, Walter de Gruyter, Berlin.

Aurell, Erik, Roberto Baviera, Ola Hammarlid, Maurizio Serva & Angelo Vulpiani (2000), 'Growth optimal investment and pricing of derivatives', *Physica A* **280**, 505–521.

Blume, Lawrence & David Easley (1992), 'Evolution and market behavior', *Journal of Economic Theory* **58**, 9–40.

Breiman, L. (1961), 'Optimal gambling for favorable games', *Fourth Berkeley Symposium on Math. Statist. and Probability* **1**, 65–78.

Browne, Sid & Ward Whitt (1996), 'Portfolio choice and the Bayesian Kelly criterion', *Advances in Applied Probability* **28**, 1145–1176.

Buchmann, Boris & Stefan Weber (2005), An evolutionary financial market - a Markovian dynamics. Working Paper, Humboldt-Universität zu Berlin and Australian National University (in preparation).

Cover, Thomas M. (1991), 'Universal portfolios', *Mathematical Finance* **1**, 1–29.

Evstigneev, Igor V., Thorsten Hens & Klaus R. Schenk-Hoppé (2002), 'Market selection of financial trading strategies: Global stability', *Mathematical Finance* **12**(4), 329–339.

Fama, Eugene F. (1965), 'The behavior of stock market prices', *Journal of Business* **38**, 34–105.

Friedman, Milton (1953), *Essays in Positive Economics*, University of Chicago Press, Chicago.

Hakansson, Nils H. (1970), 'Optimal investment and consumption strategies under risk for a class of utility functions', *Econometrica* **38**, 587–607.

Hakansson, Nils H. & William T. Ziemba (1995), Capital growth theory, *in* Robert A. Jarrow and Vojislav Maksimovic and William T. Ziemba, ed., 'Handbooks in Operations Research and Management Science – Finance', Vol. 9, Elsevier Science, chapter 3, pp. 65–86.

Hens, Thorsten & Klaus R. Schenk-Hoppé (2004), 'Evolution of portfolio rules in incomplete markets', *forthcoming in Journal of Mathematical Economics* .

Karatzas, Ioannis & Steven E. Shreve (1998), *Methods of Mathematical Finance*, Springer-Verlag, New York.

Thorp, Edward (1971), Portfolio choice and the Kelly criterion, *in* R. G. Vickson and W. T. Ziemba, ed., 'Stochastic Models in Finance', Academic Press, New York, pp. 599–619.