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Journal of Banking & Finance 28 (2004) 3009–3036

Journal of BANKING & FINANCE

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Cyclical correlations, credit contagion, and portfolio losses

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Received 12 May 2003; accepted 11 November 2003
Available online 14 January 2004

Abstract

We model aggregate credit losses on large portfolios of financial positions contracted with firms subject to both cyclical default correlation and direct default contagion processes. Cyclical correlation is due to the dependence of firms on common economic factors. Contagion is associated with the local interaction of firms with their business partners. We provide an explicit normal approximation of the distribution of portfolio losses. We quantify the relation between the variability of global economic fundamentals, strength of local firm interaction, and the fluctuation of losses. We find that cyclical oscillations in fundamentals dominate average losses, while local interaction causes additional fluctuations of losses around their average. The strength of the contagion-induced loss variability depends on the complexity of the business partner network.

JEL classification: G12; G13
Keywords: Cyclical correlation; Credit contagion; Portfolio losses; Bernoulli mixture model

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0378-4266/$ - see front matter
doi:10.1016/j.jbankfin.2003.11.002

1. Introduction

One of the long-lasting discussions in economics concerns the explanation of aggregate economic activity. In this paper we contribute to this discussion by studying the fluctuation of aggregate credit losses on large portfolios of financial positions. Our explicit results provide a number of significant insights relevant to both risk measurement and management in financial institutions and supervisory authorities.

Default rates of firms and hence credit losses vary substantially even on a high level of aggregation (see, in particular, the regular studies of the various credit rating agencies, for example Keenan (2000)). One potential explanation is easily conceived: firms' ability to generate cash flows and hence their default proneness fluctuates with the fundamentals of the economy, such as specific factor prices, demand for manufactured goods, or production costs. The dependence of firms on the general (macro-) economic environment induces dependence between firms' defaults. A high degree of such positive *cyclical default correlation* and thus a high fluctuation of aggregate default rates and credit losses would result from correlated variations in fundamental variables, which is simultaneously disastrous for a large number of firms.

In this article we investigate the economy's micro-economic structure and analyze under which conditions the effect of variations in fundamental macro-economic variables can amplify with the appearance of direct connections between firms. These direct inter-firm links are typically associated with borrowing and lending contracts or other legally binding relationships, such as parent–subsidiary structures; they provide a channel for the direct *contagion* of economic distress from one firm to other firms. A characteristic example are interbank lending agreements, which refer to banks' mutual claims. Provided that these claims are neither collateralized nor insured against, the financial distress of one institution, triggered by management failure or adverse fundamentals, may spread to several other institutions in the lending chain through default on due obligations. Such bank contagion effects are widely discussed in the micro-economic literature, see for example Allen and Gale (2000). A similar contagion mechanism is also associated with non-financial firms through the institution of trade credits, which link suppliers and buyers of goods through a chain of obligations. For a micro-economic model see, for example, Kiyotaki and Moore (1997).

Sufficiently large adverse fluctuations of default rates can lead to the distress of lending institutions. Besides being of critical importance for individual banks' risk management, the design of effective supervising policies and intervention strategies calls for a thorough understanding of the variability of aggregate losses. In this paper we study the fluctuation of aggregate credit losses on large portfolios of financial positions, taking into account both cyclical default correlation and credit contagion processes. From a methodological point of view, this provides a reconciliation of the cyclical correlation-based class of Bernoulli mixture models (Frey and McNeil, forthcoming) and an approach focusing exclusively on credit contagion based effects (Giesecke and Weber, 2002).

Bernoulli mixture models have become a standard for the measurement and management of credit loss risk in financial institutions. Examples include the models put forward by KMV (Kealhofer, 1998), J.P. Morgan (Gupton et al., 1997), Credit

Suisse Financial Products (CSFP, 1997), and McKinsey (Wilson, 1997). For an overview of these models we refer to Crouhy et al. (2000). In this class of models the fluctuation of credit losses is due to the variation of economic fundamentals only, so that firms' interdependence is related to cyclical correlation effects only. By neglecting contagion effects, such an approach might underestimate the degree of loss fluctuation to be expected. An approach focusing exclusively on the contagion effects as in Giesecke and Weber (2002), on the other hand, does not explicitly account for cyclical correlation effects. In this paper we aim at unifying these two complementary approaches. In particular we will establish an extended Bernoulli mixture model which integrates both "global" cyclical and "local" contagion effects.

We take as our starting point the model developed in Giesecke and Weber (2002), in which firms interact with their business partners in a lattice-type economy. A firm is subject to liquidity shocks from its business partners; its corresponding liquidity state is described by a binary state variable. The liquidity account can become severely "stressed" if business partners fail to honor due obligations. In this case the firm may not be able to generate sufficient cash flow to invest in production opportunities and to honor its own obligations. If the firm can buffer the adverse effects from defaulting business partners (through sufficient reserves, for example), then its liquidity state is considered "stable." We suppose that a firm migrates from one liquidity state to another with an intensity that is proportional to the number of business partners in the opposite state. The idea here is that if a stable firm's partners default on obligations, then the probability of this firm becoming liquidity stressed as well increases with the number of failing partners, since at some point the available liquidity reserves will be exhausted. Vice versa, the probability of a liquidity-stressed firm to overcome the shortage increases with the number of financially healthy partners which honor due obligations timely. The continuous-time Markov process describing the joint evolution of firms' liquidity state converges as time approaches infinity. In contrast to Giesecke and Weber (2002), we additionally describe the macro-economic business environment in the steady state by a random vector with given distribution and model the influence of both contagion and cyclical oscillations on the firms. We quantify the joint effect of these factors and characterize their relative importance.

In our model, a financial institution holds a portfolio of positions with firms in the interaction-based economy. The credit loss on a position depends on both the macro-environment and the firm's individual liquidity state resulting from the local interaction with its business partners. Our main result consists of an *explicit approximation* of the distribution of aggregate losses on a large portfolio of positions, whose issuers are subject to the macro-economic environment and credit contagion processes. This generalizes a corresponding result in Giesecke and Weber (2002), where losses are driven by contagion only. Our approximation is the key to the measurement and management of the portfolio's aggregated credit loss risk. Analogous large portfolio approximations have been proven extremely useful in the context of Bernoulli mixture models, see for example Vasicek (1987), Frey and McNeil (2002), Lucas et al. (2001), Schloegl (2002), or Gordy (2001), whose approximation results have been very influential in the design of the Basel II regulatory capital requirements. To analyze the loss distribution in comparison with that implied by a classical Bernoulli mixture model, we provide a Bernoulli mixture type representation of our model. This not only enlarges the existing Bernoulli mixture model class, but also makes estimation techniques available that are already well-known in the Bernoulli class.

Our model predicts that average losses on large portfolios of financial positions are dominated by cyclical oscillations in the economy's fundamentals. Local firm interaction and the associated contagion processes lead to additional fluctuations of losses around their averages. The existence of contagion phenomena corresponds to the presence of *additional loss risks* on the portfolio level, which cannot be attributed to the variability in the macro-economy, and which are not identified by the traditional Bernoulli mixture models. In particular, we find that the contagion effect is *relatively* large, if uncertainty about macro-economic fundamentals is low. Nevertheless, macro-economic fluctuations are the main source of loss risk. The strength of the additional contagion-induced loss variability and the probability of large losses depends on the complexity of the business partner network, i.e. the degree of connectedness between firms. Specifically, the loss variability and probability of large losses increases with decreasing complexity of business partner relations. This is in line with the predictions of the micro-economic bank contagion model of Allen and Gale (2000).

The implications of our model are consistent with a number of empirical studies. Firstly, Kaufman (1994), Upper and Worms (2002), Furfine (2003), Schoenmaker (2000), Calomiris and Mason (1997), and Elsinger et al. (2002) confirm the existence of contagion risk in the banking sector, while Lang and Stulz (1992) find contagion effects in the non-financial industry. Consistent with the predictions of our model, Calomiris and Mason (1997), Upper and Worms (2002), Schoenmaker (2000) and Elsinger et al. (2002) argue that contagion is typically only a second-order effect, dominated by systematic (macro-economic) factors. Schoenmaker (2000) finds that contagion risk can be almost eliminated by a regulator functioning as a lender of last resort. There is so far no empirical evidence on the "size" of the contagion risk relative to systematic risk, making it hard at this stage to validate our quantitative estimates on this relation.

The balance of this paper is organized as follows. In Section 2 we define an economy where firms interact with each other within a business partner network specified by a multi-dimensional lattice. We postulate contagion dynamics and analyze the long-run behavior of the firms' interaction-induced state. In Section 3 we examine credit losses due to macro-economic fluctuations as well as contagion effects. We specifically provide a normal approximation to the distribution of aggregate losses. A Bernoulli mixture type representation of our model is provided in Section 4, where we also examine the fluctuations of aggregate losses. An estimation strategy for the parameters is discussed in Section 5. Section 6 concludes and discusses some important implications of our results for the regulation of financial institutions and the control of systemic risk.

2. A statistical model of contagion

Following Giesecke and Weber (2002), in this section we set up a *statistical* model for contagion phenomena which is based on a stylistic model of local firm interaction in a *homogeneous* economy. In particular, we assume that firms have the same number of business partners and are of equal size. Nevertheless, different firms are allowed to be in different states.

2.1. Economy and firms

Our economy consists of a collection F of firms. An arbitrary firm $i \in F$ interacts with a collection $N(i) \subseteq F \setminus \{i\}$ of business partners, or *neighbors*. Typical business partners include suppliers of goods in the manufacturing process and buyers of manufactured products. The firm's creditors, such as suppliers in trade credits, banks, shareholders, or investors in the firm's public debt, as well as its borrowers (think of customers which are granted a trade credit), can also be considered as business partners. For simplicity, we assume that a firm's interaction with its neighbors is *symmetric*, in the sense that

$$j \in N(i) \Rightarrow i \in N(j). \tag{1}$$

Put another way, if a firm $j \in F$ is the business partner of another firm $i \in F \setminus \{j\}$, then i is also a business partner of j. If we connect all firms $i \in F$ with their neighbors $j \in N(i)$, we get an undirected graph which characterizes the business relations of the firms. For tractability, however, we shall assume a simple neighborhood structure and identify firms with their location on the d-dimensional integer lattice $F = \mathbb{Z}^d$. On this lattice the distance between two firms determines whether they are business partners. For concreteness, we define the neighborhood $N(i)$ of a firm i by

$$N(i) = \{j : |j - i| = 1\}, \tag{2}$$

where $|\cdot|$ denotes the length of the shortest path between two firms on the lattice. In other words, two firms are business partners if the shortest distance between them is one unit on the lattice. Fig. 1 illustrates this in the two-dimensional case $d = 2$.

The dimension d of the lattice can be interpreted as the degree of complexity of the business partner network. With increasing d the structure of inter-firm connections becomes more complex. The larger d, the more business partners has any individual firm. At the same time the number of indirect inter-firm links of given length increases.

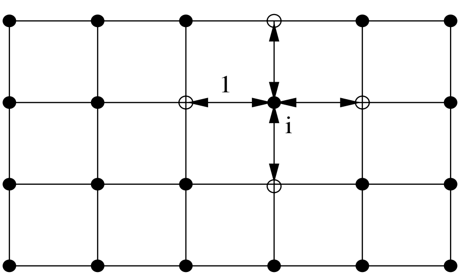

Fig. 1. A sector of the lattice economy in case $d = 2$. Every firm $i \in \mathbb{Z}^2$ interacts with four business partners $j \in N(i)$.

Having defined the structure of our interaction-based economy, let us now consider the process of interaction in more detail. In the business partner network firms are linked through a chain of obligations. If one link in this chain does not honor obligations timely or defaults entirely on some of them, then this immediately reduces the amount of liquidity available to this firm's business partners. We can here think, for example, of a temporary liquidity shortage leading to a payment default of a buyer in a trade credit in the first place. Supposing that the immediate liquidation value of the underlying goods used as collateral is sufficiently low, then this default may eventually result in the supplier becoming short of liquidity as well. Given the lack of liquidity, the supplier may be prevented from investing in production opportunities and realizing the associated returns in the future. This reduction in the supplier's cash flow ability may lead to the supplier defaulting on obligations with other business partners as well. We refer to this process as *credit contagion*.

We establish a simple probabilistic model for the dynamics of credit contagion over time. To this end, we associate with each firm $i \in \mathbb{Z}^d$ a state variable $\xi(i) \in \{0, 1\}$, which describes the firm's liquidity state with respect to the interaction with its business partners $N(i)$. State $\xi(i) = 1$ means that firm i's liquidity reserves are stressed and might be insufficient to honor due obligations. State $\xi(i) = 0$ means that firm i is financially healthy and honors its obligations to business partners timely.

Firms are subject to liquidity shocks in positive and negative direction. It is hence natural to assume that a transition of firm i from liquidity state $\xi(i)$ to state $1 - \xi(i)$ is an unpredictable Poisson event. The stochastic structure of this event is described by an intensity. We suppose that this transition intensity is proportional to the number $|\{j \in N(i) : \xi(j) = 1 - \xi(i)\}|$ of business partners in the opposite state. [2] The proportionality factor [3] is normalized to $\frac{1}{2d}$. An equivalent way to describe these dynamics is as follows. After a unit-exponential waiting time, a firm i adopts the liquidity state of one of its $2d$ business partners which is chosen with uniform probability $\frac{1}{2d}$. In the two-dimensional case illustrated in Fig. 1, i has four partners, and it adopts the state of one of them chosen with probability $\frac{1}{4}$. Of course, the more partners are in opposite liquidity state, the higher is the probability that i changes its state. Formally, the evolution of firms' liquidity state over time is modeled by a continuous-time Markov process $(\eta_t)_{t \geqslant 0}$ with state space $X = \{0, 1\}^{\mathbb{Z}^d}$ and transition rate c given by

$$c(i, \xi) = \begin{cases} \frac{1}{2d} \sum_{j \in N(i)} \xi(j) & \text{if } \xi(i) = 0, \\ \frac{1}{2d} \sum_{j \in N(i)} [1 - \xi(j)] & \text{if } \xi(i) = 1. \end{cases} \tag{3}$$

[2] The idea that a firm's state depends on the state of other firms in the economy has recently appeared in Jarrow and Yu (2001) and Davis and Lo (2001). Using the concept of a default intensity, these authors focus directly on the default state in its relation to other firms, whereas we consider the liquidity state of a firm. Davis and Lo (2001), for example, suppose that after a default in the market, the default intensities of remaining firms are increased for an exponential time, before falling back to their normal levels.

[3] The choice of the proportionality factor does not have any influence on the time asymptotics of the system. Changing this factor is equivalent to a linear deterministic time change.

This is known as the basic voter model in the theory of interacting particle systems (Liggett, 1985, 1999).

Assumption (3) captures the idea that liquidity shocks can be propagated: If a currently liquidity-stable firm's business partners default on obligations, then the probability of this firm becoming liquidity stressed as well increases with the number of failing partners, since at some point the available liquidity reserves will be exhausted. Vice versa, the probability of a liquidity-stressed firm to overcome the shortage increases with the number of financially healthy business partners which honor due obligations timely.

Implicit in our simple probabilistic contagion model (3) are symmetry assumptions. The first is related to the direction in which shocks are propagated. As just described, a healthy firm can become distressed through shock propagation. On the other hand it is also realistic that a stressed firm can make a turnaround due to positive liquidity shocks form partners. [4] Less realistic is that business partner relationships are symmetric in the sense of (1). Indeed, think of a small firm supplying exclusively to a big automobile manufacturer: while low liquidity of the manufacturer can be disastrous for the supplier, low liquidity of the supplier has a less severe impact on the manufacturer. These effects could be modeled in the context of interacting particle systems on directed graphs.

Several assumptions that we made along the way can be substantially weakened at the cost of added complexity. One is to generalize our narrow neighborhood definition (2) so as to allow for interaction of a given firm i with any other given firm j according to the transition probability $p(i,j)$ of a random walk on $\mathbb{Z}^d$. We would then substitute (3) with

$$c(i,\xi) = \begin{cases} \sum_j p(i,j)\xi(j) & \text{if } \xi(i) = 0, \\ \sum_j p(i,j)[1-\xi(j)] & \text{if } \xi(i) = 1. \end{cases}$$

All the results we develop below go through at this level of generality.

2.2. Convergence to equilibrium

We are interested in the asymptotical behavior [5] of the Markov liquidity state process η_t as $t \to \infty$ and the equilibrium distributions of firms' liquidity state. Throughout, we let μ denote the initial distribution of η, which we may think of arising from general (macro-) economic conditions. We assume that μ is translation-invariant and denote by

$$\rho = \mu\{\xi : \xi(i) = 1\} \tag{4}$$

[4] In order to keep the analysis simple, we assume that the strength of the influence of its neighbors is the same for healthy and stressed firms. An example for a model with non-symmetric interaction is the *contact process*.

[5] When analyzing the time asymptotics of the process η we use the notion of *weak convergence* of probability measures.

the probability under μ that an arbitrary firm i is liquidity stressed. In particular, the translation-invariance of μ implies that the firms in our economy are *homogeneous* with respect to ρ. While ρ is invariant under the contagion dynamics, the economy can change drastically on the macroscopic level. The structure of the equilibrium distributions depends on the complexity of the business partner network.

For a simple connectivity structure with $d = 1, 2$, the process η *clusters*: in the long run $t \to \infty$ the economy ends up in one of two possible extreme scenarios. Asymptotically, with probability ρ *all* firms are liquidity stressed, and with probability $1 - \rho$ *all* firms are stable. Let us suppose that initially ρ is high, so that the probability that an individual firm is liquidity stressed is high. Then random clusters of liquidity-stressed firms emerge with high probability, while clusters of stable firms emerge only with low probability. The size of the clusters changes through random fluctuations, but some of the clusters merge and form large growing clusters. In the long run the entire economy is a single cluster of firms of the same type. Since ρ is large, the probability that *all* firms are finally liquidity stressed is high.

The limiting behavior of the Markov process η differs for higher dimensions $d > 2$. Here in the long run the process η *coexists*, meaning that heterogeneity in firms' states will appear for $\rho \in (0, 1)$. Random clusters of firms of equal state appear here only locally; they do not persist and do not grow in the same way we observed with $d = 1, 2$. The equilibrium distribution of η_t for $t \to \infty$ is given by the mixture

$$\int_{[0,1]} v_\rho Q(\mathrm{d}\rho), \tag{5}$$

cf. Giesecke and Weber (2002). Here, v_ρ is the extremal invariant measure of the basic voter model in dimension $d > 2$ with parameter $\rho \in [0, 1]$ (cf. Liggett, 1999), and Q is the distribution of the empirical average of liquidity-stressed firms in the whole economy:

$$\lim_{n\to\infty} |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \xi(i) = \bar{\rho}, \tag{6}$$

where $\Lambda_n = [-n, n]^d$. In general, $\bar{\rho}$ is not deterministic, but *random*. The empirical average of the number $\bar{\rho}$ of firms in state 1 and its distribution Q are invariant under the contagion dynamics. Nevertheless, interaction between firms strongly effects the correlation between the states of different firms. For any finite number of firms, the probability to find many firms in the same state is higher than in the case of independent firms.

3. Portfolio losses

In this section we examine the fluctuation of aggregate losses on a portfolio of debt contracts written by firms subject to the contagion processes described in the previous section. We assume that inter-firm connections are complex ($d > 2$). Throughout, we suppose that the economy is in a steady equilibrium state, in the sense that the distribution of firms' interaction-based state is invariant. The steady state distribution of the liquidity configuration will be denoted by

$$\mu = \int_{[0,1]} \nu_\rho Q(\mathrm{d}\rho). \tag{7}$$

We consider a financial institution holding a *portfolio* $\Lambda_n = [-n, n]^d$ of financial positions contracted with the firms in the interaction-prone economy. The parameter $n \in \mathbb{N}$ determines the portfolio's size, i.e. the number $\Lambda_n = (2n+1)^d$ of firms in the portfolio. The market value of a portfolio position is subject to the credit quality of its issuer or counterparty. Such positions can include not only loans, bonds, or other debt instruments, but also derivatives written by default-prone counterparties. Due to adverse changes in a counterparty's credit quality the market valuation of the corresponding positions can be severely reduced. Risk measurement aims at evaluating the potential losses induced by credit quality deterioration of firms in portfolio Λ_n. Denoting the losses on positions contracted with firm $i \in \Lambda_n$ by the random variable $U(i)$, we are thus interested in the distribution of *aggregated portfolio losses*

$$L_n = \sum_{i \in \Lambda_n} U(i). \tag{8}$$

In Section 3.1 below we model the probabilistic properties of the position losses, which then allows us in Sections 3.2 and 3.3 to study the distribution of aggregate losses L_n in detail.

3.1. Position losses

The loss $U(i)$ the financial institution incurs from positions contracted with firm $i \in \Lambda_n$ will depend on the credit quality of the firm, i.e. its ability to generate the required cash flows in the future. In our model, this cash flow ability is not only determined by the firm's liquidity state resulting from the interaction with its business partners, but also by the state of the general macro-economy (the business environment) in which the firms operate. In this sense both "global" business cycle fluctuations and "local" interaction-induced contagion processes corresponding to the economy's micro-firm structure affect credit losses. This is in fact an important conceptual advancement over the existing Bernoulli mixture models (Frey and McNeil, forthcoming), which have become a standard for credit risk measurement and management in financial institutions. The key assumption of Bernoulli mixture models is that credit losses are conditionally independent given the macro-economic state. Dependence between losses on firms' positions arises through the dependence of firms on the common macro-state variables. Here firms' interdependence is related to cyclical correlation effects only; effects stemming from direct firm interaction are not captured. Our model, in contrast, captures both cyclical and contagion effects.

For the specification of the probabilistic properties of the position losses $U(i)$ we note first that for the credit contagion process being in some tuned steady equilibrium state, we know the joint distribution (7) of firms' interaction-induced liquidity state. The state of the general (macro-) economy prevailing in the steady state will then be described by some random vector $K \in \mathbb{R}^p$ with given distribution κ. The vector K is common to all firms and captures the economy's business environment. Elements of K may include indicators of the stage of the business cycle, commodity and factor prices, inflation rates, or interest rates.

Now our key assumptions on the probabilistic structure of the $U(i)$ are as follows. Conditional on the macro-economic state $K \in \mathbb{R}^p$ and the liquidity profile $\xi \in \{0,1\}^{\mathbb{Z}^d}$, losses are independent. The conditional loss distribution of a firm is denoted by $M_{k;x}$; it depends only on the economy-wide macro-economic state k and the firm-specific liquidity state x. [6] Given k and x we denote the expected losses conditional on these states by $l_x(k) := \int u M_{k,x}(\mathrm{d}u)$, and assume that for every $k \in \mathbb{R}^p$ the inequality $l_0(k) < l_1(k)$ holds.

All firms in the same liquidity state respond to systematic risk in the same way. This homogeneity is common to most Bernoulli mixture models; for a notable exception see Pesaran et al. (2003). We could generalize to heterogeneous responses, at the cost of added complexity. We do not pursue this here.

3.2. Distribution of portfolio losses

Individual position losses are not independent but coupled through both the macro-economic factor and the interactive liquidity state. When analyzing portfolio losses, it is important to respect the resulting dependence structure. The induced joint distribution of the losses of all firms is a probability measure on $\mathbb{R}_+^{\mathbb{Z}^d}$. Due to direct firm interaction this measure cannot be described as a mixture of product measures as in the case of macro-economic dependence only. Instead, also the local dependence of firms' losses must be captured. Using non-standard techniques we will be able to analyze the joint distribution of the losses which will be denoted by β. [7]

The distribution β captures the probability of events specified in terms of individual losses for a *collection* of firms. For example, assume that $i, j \in \mathbb{Z}^d$ are two different firms in our economy, and a_i, a_j are two positive real numbers. We define the cylinder set $A \subseteq \mathbb{R}_+^{\mathbb{Z}^d}$ by

[6] For technical reasons, we assume that the mapping

$$\begin{cases} \mathbb{R}^p \times \{0,1\} & \rightarrow \quad \mathscr{M}_1(\mathbb{R}_+), \\ (k,x) & \mapsto \quad M_{k,x}, \end{cases}$$

is measurable. Here, $\mathscr{M}_1(\mathbb{R}_+)$ denotes the space of Borel probability measures on the positive real line with the weak topology. Moreover, we will suppose that all measures $M_{k,x}$ with $(k,x) \in \mathbb{R}^p \times \{0,1\}$ are supported in a common interval $[0,c]$ for some $c > 0$.

[7] Recall that κ is the distribution of macro-economic factors and that μ given by (7) governs firms' equilibrium liquidity configuration. In terms of mixtures of the position loss distributions $M_{k,x}$ the firms' joint loss distribution β can be written as

$$\beta(\mathrm{d}w) = \int\int (\otimes_{i \in \mathbb{Z}^d} M_{k,\xi(i)})(\mathrm{d}w)\mu(\mathrm{d}\xi)\kappa(\mathrm{d}k), \quad w \in \mathbb{R}_+^{\mathbb{Z}^d}.$$

$$A = \{w \in \mathbb{R}_+^{\mathbb{Z}^d} : w_i \leqslant a_i, w_j \leqslant a_j\}.$$

Then the probability of a loss on a position contracted with i of less than a_i occurring together with a loss on a position contracted with j of less than a_j is given by

$$\text{Prob}(U(i) \leqslant a_i, U(j) \leqslant a_j) = \beta(A).$$

We are interested in the distribution of aggregated losses L_n under the measure β. For large n, we will approximate both the average loss in the portfolio Λ_n and the fluctuations of the losses around their averages. Mathematically, approximate average losses can be characterized by a *law of large numbers*. Fluctuations around averages are approximated using a *non-classical central limit theorem*.

Average losses. We begin with the investigation of average losses. If we normalize portfolio losses L_n by the number of positions $|\Lambda_n|$ in the portfolio, we obtain convergence to a random variable as $n \to \infty$. That is, by the law of large numbers we obtain convergence to a random variable β-almost surely,

$$\lim_{n\to\infty} \frac{L_n}{|\Lambda_n|} = \bar{\rho} \cdot l_1(K) + (1 - \bar{\rho}) \cdot l_0(K). \tag{9}$$

The limit on the right hand side of (9) is well-understood. K denotes the macro-economic factor, $\bar{\rho}$ the average number of firms in the whole economy in state 1. The random variables K and $\bar{\rho}$ are independent and distributed according to κ and Q, respectively.

Conditional on $K = k$ and $\bar{\rho} = \rho$ for some constants $k \in \mathbb{R}^p$ and $\rho \in [0, 1]$, average losses are a convex combination of the conditional expected losses $l_x(k)$ given the macro-economic factor k and the liquidity state $x \in \{0, 1\}$. The weights in (9) are the proportions ρ and $1 - \rho$ of firms in the economy in liquidity state 1 or 0. It appears that in the limit all loss fluctuations due to the randomness in the conditional position losses $M_{k,x}$ are averaged out; only their expectations $l_x(k)$ given the interaction state $x \in \{0, 1\}$ and the macro-factor $k \in \mathbb{R}^p$ enter average losses.

Fluctuation of losses. In order to get a more detailed picture about the fluctuation of losses and their relation to the state of the economy and the interaction between firms, we will now provide a Gaussian approximation of the distribution of aggregate losses using a non-classical central limit theorem. To this end we introduce the functions

$$\tilde{l} : \begin{cases} \mathbb{R}^p & \to & \mathbb{R}_+^2, \\ k & \mapsto & (l_0(k), l_1(k)). \end{cases}$$

The first component of $\tilde{l}$ is the conditional expected loss of a single firm, if it is in liquidity state 0 and the macro-factor equals k; the second component is the expected loss conditional on the liquidity state 1 and macro-factor k. Hence, we will call $\tilde{l}$ the *conditional expected loss vector*.

Letting Φ denote the standard normal distribution function, we will show in Theorem 3.1 below that the loss distribution can be approximated using the function

$$\psi_{d,a}(r;\rho,l_0,l_1)=\Phi\left(\frac{r\cdot m(\rho,l_0,l_1)-a}{(l_1-l_0)\sigma(d,\rho)r^{(d+2)/2d}}\right),\quad a>0,\ \ r>0. \tag{10}$$

Here

$$m(\rho,l_0,l_1)=\rho\cdot l_1+(1-\rho)\cdot l_0 \tag{11}$$

and $\sigma^2=\sigma^2(d,\rho)$ is a constant given in Appendix A. The argument $r>0$ denotes the number of firms in the portfolio.

We are now ready to approximate the distribution of aggregate losses L_n for large portfolios. This approximation is the key to the measurement and management of the portfolio's aggregated credit loss risk. Analogous large portfolio approximations have been proven extremely useful in the context of Bernoulli mixture models, see for example Vasicek (1987), Frey and McNeil (2002), Lucas et al. (2001), or Schloegl (2002). Our approximation result covers not only the cyclical effects the Bernoulli mixture models are concerned with, but also direct contagion effects. The proof is given in Appendix A.

Theorem 3.1. *Let $d>2$, and assume that $Q(\{0\})=Q(\{1\})=0$. For a portfolio of size $r>0$ and a lower bound for the losses of $a>0$ we define the function*

$$\Psi_{d,a}(r)=\int\int\psi_{d,a}(r;\rho,l_0,l_1)Q(\mathrm{d}\rho)F(\mathrm{d}(l_0,l_1)), \tag{12}$$

where $F=\kappa\circ\tilde{l}^{-1}$ is the law of the conditional expected loss vector $\tilde{l}$ under the macro-factor distribution κ. For a large portfolio, the distribution of losses L_n can be uniformly approximated:

$$\sup_{a\in\mathbb{R}}|\beta(L_n\geqslant a)-\Psi_{d,a}(|\Lambda_n|)|\leqslant\epsilon_n,$$

where the error $\epsilon_n\to 0$ as $n\to\infty$.

By definition, $\beta(L_n\geqslant a)$ is the probability of a loss larger than $a>0$ in the portfolio containing $|\Lambda_n|$ firms. By Theorem 3.1, the function $\Psi_{d,a}(|\Lambda_n|)$ approximates this probability uniformly in the loss level a. Heuristically, we may interpolate between sizes of portfolios and replace $|\Lambda_n|$ by some real number $r>0$. Then we will call $\Psi_{d,a}(r)$ the approximate probability for a loss larger than $a>0$ on a portfolio of size $r>0$.

Let us now analyze the function $a\mapsto\Psi_{d,a}(r)$. The approximate loss probability has the following properties which are apparent from (10) and (12):

- The approximate loss probability is a mixture of Gaussian probabilities.
- The individual Gaussian probabilities do not have variance of order $r^{1/2}$ as in the case of r independent firms. Instead, the order of the variance is larger and equal to $r^{(d+2)/2d}$. This is due to the interaction of firms.
- The exponent $\frac{d+2}{2d}$ decreases, if d increases, and converges to $\frac{1}{2}$ when $d\to\infty$.
- The mixture is governed by the distribution Q of the average number of firms with low liquidity and the distribution F of the conditional expected loss vector.

- Randomness in the conditional position losses $M_{k,x}$ is averaged out; only the conditional expected loss vector $\tilde{l}$ matters.

Since the individual Gaussian probabilities have a *variance* of a larger order in r than in the standard case, the risk of large losses is increased. For small d this effect can actually be quite substantial. In the next section we will investigate how its strength depends on the properties of the macro-fluctuations. As a mixture of Gaussian distributions the approximate loss distribution may have a *fat-tailed*, non-Gaussian shape. This will be caused by fluctuations due to the distributions Q and κ.

3.3. Contagion versus cyclical correlation

The shape of the approximate loss distribution depends on the size of the contagion parameter d and on the mixing distributions Q and F. Contagion induces fluctuations of the losses in large portfolios around their random means. The means are determined by the macro-economic factor K and the average number of liquidity-stressed firms $\bar{\rho}$. In this section we will investigate how the approximate loss distribution is influenced by both interaction and macro-economic fluctuations.

Macro-economic uncertainty is described by the distribution κ. A priori it is not clear how different components of this "risk" affect the loss distribution. In order to identify the relevant (macro-economic) loss drivers, we re-parameterize the model. We introduce the variables m, Δ, ρ where

$$m = \rho \cdot l_1 + (1 - \rho) \cdot l_0, \tag{13}$$

$$\Delta = l_1 - l_0. \tag{14}$$

The variable m describes the average loss in the whole economy which is almost surely defined by the limit

$$\lim_{n\to\infty} \frac{L_n}{|\Lambda_n|} = m. \tag{15}$$

Δ is given by the difference between the expected losses $l_1(k)$ and $l_0(k)$, and ρ equals the proportion of liquidity-stressed firms in the economy. These quantities describe different aspects of the systematic risk in relation to losses. We will investigate the response of the distribution of aggregated losses to these components.

The distribution κ of the macro-economic factor and the distribution Q of the average liquidity state induce a joint distribution of the triple (m, Δ, ρ). For simplicity, we assume that Q is a Dirac measure concentrated on $\frac{1}{2}$, meaning that always 50% of all firms in the economy are liquidity stressed. We remark that the influence of the distribution Q on the contagion effect resembles qualitatively the implications induced by the distribution of m which we will discuss in detail below. We denote the joint image distribution of (m, Δ) by $v(\mathrm{d}m, \mathrm{d}\Delta)$. From (12) we obtain the following result.

Corollary 3.2. *Under the simplifying assumption $Q = \delta_{1/2}$, the approximate probability of a loss larger than $a \in \mathbb{R}_+$ for a portfolio of size $r > 0$ is given in terms of the new parameters by the expression*

$$\Psi_{d,a}(r) = \int \Phi\left(\frac{r \cdot m - a}{\Delta \cdot \sigma(d, \rho) \cdot \sqrt{r} \cdot r^{1/d}}\right) v(\mathrm{d}m, \mathrm{d}\Delta). \tag{16}$$

Qualitative implications of Corollary 3.2 are summarized in Table 1. These results will be derived from several case studies in the remaining part of this section. The effect of contagion on the shape of the approximate loss distribution mainly depends on the contagion parameter d and on the uncertainty about average losses in the whole economy m.

The quantity d measures the degree of complexity of the business partner network. Specifically, with increasing complexity d the contagion effect decreases and the probability of the large losses becomes smaller. This relation is consistent with the micro-economic bank contagion model of Allen and Gale (2000), for example; it was recently empirically confirmed by Upper and Worms (2002). Allen and Gale (2000) considered an interbank market structure which can be complete (any bank is connected to any other bank through deposits) and incomplete (a given bank is only connected with a limited number of other banks, but not all of them). They found that contagion was not an issue in the complete market structure. With the incomplete structure the likelihood of contagion can be substantial, and depends on how "well connected" banks are. This is consistent with our model, which is of Allen and Gale's (2000) incomplete structure type.

The distribution of average losses in the economy m is the relevant statistic that captures the qualitative influence of macro-economic fluctuations on the portfolio losses. The contagion effect becomes relatively large, if uncertainty in m is reduced. If the fluctuations of average losses m are large due to high macro-economic uncertainty, the contagion effect is less significant.

Case studies. To derive the qualitative implications of Corollary 3.2, we consider several representative case studies. We investigate the contagion effect for different specifications of the measure v, namely the three cases:

(a) Δ fixed, m random;
(b) Δ random, m fixed; and
(c) Δ, m both random.

Table 1
Strength of the contagion effect

	Contagion effect	
	Large	Small
Distribution of m	Concentrated	Spread out – large uncertainty
Connectedness d	Low connectedness – small d	Strong connectedness – large d

The quantity m describes the average loss per firm in the whole economy. Its distribution captures the systematic risk related to the macro-economic factor. The quantity d measures the degree of complexity in the business partner network.

For each of the choices for the distribution of v we will compare the contagion effect for degrees of connectedness of the economy equal to $d = 3, 4, 5$. If $Q = \delta_{1/2}$, the uncertainty about m and Δ is completely governed by the distribution of the macroeconomic factor. Hence, the distributions of both m and Δ capture the systematic risk in the economy.

We focus first on the case (a) in which m is the only random parameter. The contagion effect depends crucially on the shape of the distribution of m. By assumption (a) the distribution v equals a product measure. Letting e.g. $\Delta = 0.5$, v is thus given by

$$v(\mathrm{d}m, \mathrm{d}\Delta) = \chi(\mathrm{d}m) \otimes \delta_{0.5}(\mathrm{d}\Delta). \tag{17}$$

The contagion effect depends on the uncertainty about the parameter m which is modeled by the distribution χ. We investigate three cases:

(1) *No uncertainty about m*: The distribution of m is close to a Dirac measure.
(2) *Different regimes for m*: Average losses m can only take values in a few different small parameter regions, but the correct value is unknown. That is, the distribution of m is close to a mixture of Dirac measures.
(3) *Large uncertainty about m*: The distribution of m is close to a uniform distribution on an interval of considerable size.

In Figs. 2–4, we plot the approximate loss densities for the interaction cases $d = 3, 4, 5$ for a portfolio of size $r = 10\,000$ choosing

$$\chi = \delta_{0.5}, \quad \chi = \frac{1}{5} \cdot (\delta_{0.3} + \delta_{0.4} + \cdots + \delta_{0.7}), \quad \chi = \mathrm{unif}[0.3, 0.7],$$

respectively. These choices are caricatures of the three cases (1)–(3) above.

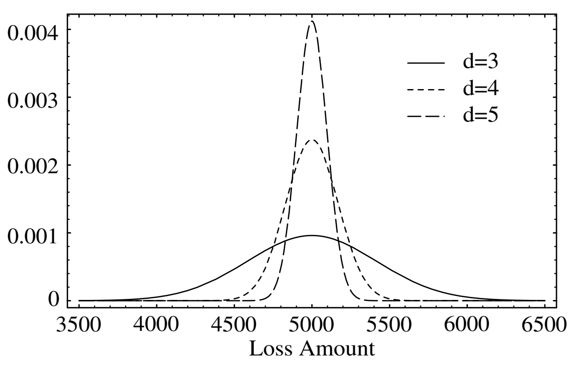

Fig. 2. Approximate loss density, varying the degree of connectedness d ($r = 10\,000$, $\rho = 0.5$, $\Delta = 0.5$, and $\chi = \delta_{0.5}$).

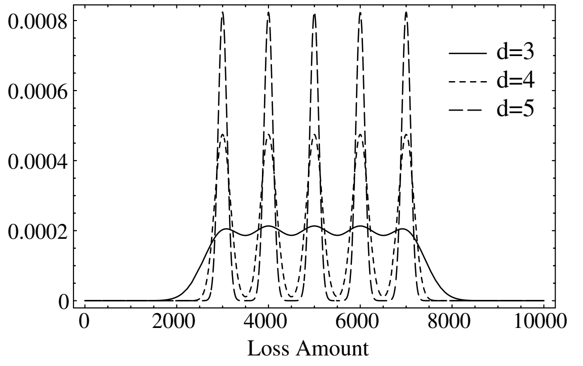

Fig. 3. Approximate loss density, varying the degree of connectedness d ($r = 10\,000$, $\rho = 0.5$, $\Delta = 0.5$, and $\chi = \frac{1}{5}(\delta_{0.3} + \delta_{0.4} + \cdots + \delta_{0.7})$).

For $\chi = \delta_{0.5}$ (Fig. 2) portfolio losses fluctuate around their mean $r \cdot m = 5000$. For a low degree of connectedness d losses fluctuate more excessively when compared with higher values of d. This is due to the variance being of order $r^{(d+2)/2d}$. The difference of the contagion effect for varying degrees of interaction d is quite significant in the case in which the support of χ is small. Observe that in terms of loss probabilities the relative size of the contagion effect equals the difference of the areas below the density curves. [8]

In Fig. 3 we illustrate the contagion effect for χ being a convex combination of Dirac measures. In this case strong interaction of firms corresponding to low values of d induces additional fluctuations around the random means $r \cdot \rho = 3000, 4000, \ldots, 7000$ – leading to a smoother loss density with less prominent peaks. At the same time probabilities of large losses slightly increase when d decreases, but this effect is less significant than in Fig. 2. Observe that the ordinate axis in Fig. 3 is differently scaled than in Fig. 2. Hence, probabilities of large losses change less when varying d – the size of the contagion effect is considerably smaller.

The approximate loss densities for $\chi = \mathrm{unif}[0.3, 0.7]$ are shown in Fig. 4. For χ having large support, the increase of probabilities of large losses due to contagion is not very strong, and the approximate loss distributions do not differ much for various degrees of interaction. We emphasize again that the ordinate axis in Fig. 4 is differently scaled than in Figs. 2 and 3. The difference of the areas below the density curves is smallest for χ being uniform.

[8] To be more precise: for a given level $a > 0$ the probability of a loss larger than a on a portfolio is influenced by the contagion parameter d. Let us denote by f_d the approximate loss density for given d. Then the approximate probability of a loss larger than a is given by the area under the density curve, namely by $\int_a^\infty f_d(x)\,\mathrm{d}x$. A quantitative measure for the relative size of contagion for different values of d and a fixed level $a > 0$ is thus given by the difference between the areas to the right of a below the density curves. This quantity is simply equal to the difference of the probabilities of approximate losses larger than a for different values of d.

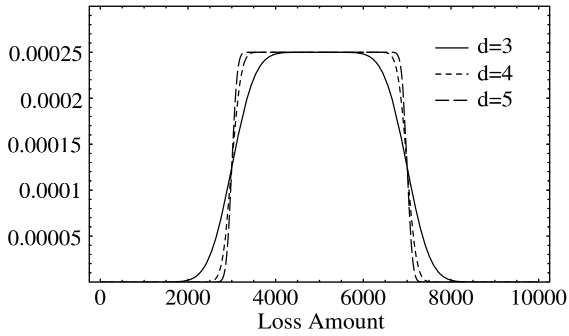

Fig. 4. Approximate loss density, varying the degree of connectedness d ($r = 10\,000$, $\rho = 0.5$, $\Delta = 0.5$, and $\chi = \text{unif}[0.3, 0.7]$).

The properties of the distribution χ influence the size of the contagion effect significantly. Let us now look at case (b) investigating the relationship between Δ and the shape of the loss distribution. It is apparent from (16) that Δ is a factor in the denominator of the argument of the cumulative normal distribution function. Hence, multiplying Δ by a constant factor is equivalent to rescaling the difference of the losses from the mean $r \cdot m$ by a constant factor. For both low and high degree of interaction the fluctuations around $r \cdot m$ are multiplied by the same factor. If Δ is random, loss distributions are simply superpositions of normal variables with mean $r \cdot m$. Nevertheless, contagion effects are qualitatively the same for non-random and random Δ; i.e. for low degrees of connectedness fluctuations around the mean $r \cdot m$ are more excessive than for high degrees of connectedness.

If m and Δ are both random as in case (c), the size of the contagion effect depends on the marginal distribution of m. If the marginal distribution of m is close to a uniform distribution, the contagion effect is small. Conversely, if the marginal distribution of m is dominated by peaks, contagion smoothes the loss distribution. If the marginal distribution of m has small support, the contagion effect is strongest. While the contagion effect is determined by the marginal distribution of m, the actual shape of the loss distribution for given contagion parameter d is governed by the joint distribution of m and Δ. Observe finally that for given d approximate loss distributions can be very similar for different specifications of the joint distribution of m and Δ; nevertheless, if the marginal distributions of m differ considerably, the size of the induced contagion effects might be quite different when varying d.

4. Bernoulli mixture representation

The class of existing Bernoulli mixture models has proven to be useful in practice to model loss distributions in the context of cyclical correlations between losses on individual portfolio positions. In this section we provide a Bernoulli mixture type specification of our model, and thereby enlarge the existing Bernoulli mixture class with a model accommodating *both* cyclical and contagion effects. We will demonstrate how the joint influence of cyclical fluctuations and contagion can be analyzed in the context of Bernoulli mixture models using the methods discussed in the last section.

Bernoulli mixture models. The key to the Bernoulli mixture type representation lies in a particular specification of the conditional distribution $M_{k,x}$ of position losses $U(i)$ for a firm $i \in \Lambda_n$ in interaction-based liquidity state $x \in \{0,1\}$ when the state of the macro-economy is $k \in \mathbb{R}^d$. We put

$$M_{k,x} = \begin{cases} 1 & \text{with probability } P_x(k), \\ 0 & \text{with probability } 1 - P_x(k), \end{cases} \tag{18}$$

so that, conditionally on (k, ξ), the position loss is a Bernoulli random variable with parameter $P_x(k)$, and we have for the conditional expected loss vector $\tilde{l}(k) = (P_0(k), P_1(k))$. The probability $P_x(k)$ is supposed to depend on x and k in a measurable way. We can interpret $P_x(k)$ as the probability of default for a firm in interaction state x when the economy is in state k, which results in a loss of one unit of account. [9]

Given the generalized Bernoulli mixture representation of our model, we can take advantage of the existing estimation models for the mixing distribution. Some of these models are outlined in the examples below. Note that in contrast to the standard models, here the default probability does depend on the macro-factor *and* the interaction-based liquidity state x.

Example 4.1. (Bernoulli regression model). Let $F : \mathbb{R} \to [0,1]$ be some strictly increasing continuous function, and let α_1, α_2, α_3 be regression parameters with $\alpha_2 > 0$. We let $K \in \mathbb{R}$ be one-dimensional with given distribution and put

$$P_x(K) = F(\alpha_1 K + \alpha_2 x + \alpha_3).$$

For different choices of the regression relationship we refer to Joe (1997). The one-factor regression model may not be flexible enough; we can generalize to higher dimensions of the macro-factor vector K. In this case, α_1 must simply be replaced by an appropriate parameter vector. Specific models of such type are provided by the following examples. We start with a choice corresponding to the CreditRisk+ model structure.

Example 4.2. (Gamma model). Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}_+^p$, and $\gamma_1, \gamma_2 > 0$ be factor weights. Let $K \in \mathbb{R}^p$ be a p-dimensional iid-Gamma vector and put

$$P_x(K) = 1 - \exp\left(- \sum_{i=1}^{p} \alpha_i K_i - \gamma_1 x - \gamma_2 \right).$$

[9] As discussed in Section 3.1, with respect to the position losses all firms respond to systematic risk in the same way. Hence, in the context of Bernoulli mixture models the factor loadings are the same for all firms. This is consistent with the majority of Bernoulli mixture models proposed in the literature.

Example 4.3. (Probit normal model). Let $\alpha = (\alpha_1, \dots, \alpha_p)$, $\gamma_1 > 0$, and γ_2 be factor weights. Let $K \in \mathbb{R}^p$ be a p-dimensional normally distributed random vector and put

$$P_x(K) = \Phi\left(-\sum_{i=1}^{p} \alpha_i K_i + \gamma_1 x + \gamma_2 \right).$$

This model parallels the choices of KMV and CreditMetrics. The assumption of normality of the factors is not essential; other distributions such as the t-distribution or more general mean–variance mixtures are possible, see Frey and McNeil (forthcoming). The following specification is similar in spirit to the CreditPortfolioView model.

Example 4.4. (Logit normal model). Let $\alpha = (\alpha_1, \dots, \alpha_p)$, $\gamma_1 > 0$, and γ_2 be factor weights. Let $K \in \mathbb{R}^p$ be a p-dimensional normally distributed random vector and put

$$P_x(K) = \left(1 + \exp\left(-\sum_{i=1}^{p} \alpha_i K_i + \gamma_1 x - \gamma_2 \right) \right)^{-1}.$$

In the context of a specific example, we now investigate the approximate loss distribution (12) for the specification (18) of our general model under various assumptions on the dependence between firms. This allows us to evaluate the effects of cyclical default correlation and credit contagion on the fluctuation of aggregate losses in the context of the Bernoulli mixture model class. We demonstrate that the methods introduced in the previous section can successfully be applied to generalized Bernoulli mixture models. The calibration of the models in practice is, of course, an empirical issue; an estimation strategy will be provided in Section 5.

A case study. As in Section 3.3, we assume for simplicity that the average proportion (6) of firms in liquidity state 1 is equal to the constant ρ, i.e. $Q = \delta_p$. In order to demonstrate how the joint influence of cyclical correlations and interaction can be analyzed using the results of Section 3.3, we choose as an example a one-factor version of the Probit normal model of Example 4.3 for the default probability $P_x(K) = l_x(K)$, which parallels the models of KMV and CreditMetrics:

$$P_x(k) = \Phi(-\alpha k + \gamma_1 x - \gamma_2). \tag{19}$$

We set $\alpha = 1$, $\gamma_1 = 2$, and $\gamma_2 = 3$, and consider different choices for the distribution of the macro-economic factor K.

Under our current assumptions, for a portfolio of size $r > 0$ the function (12) uniformly approximating the probability of aggregate losses being larger than $a > 0$ becomes

$$\Psi_{d,a}(r) = \int \Phi\left(\frac{\sqrt{r} m(\rho, l_0(k), l_1(k)) - a/\sqrt{r}}{(l_1(k) - l_0(k))\sigma(d, \rho) r^{1/d}} \right) \kappa(\mathrm{d}k), \tag{20}$$

where $l_x(k)$, m, and σ are given by (19), (11) and (A.1), respectively. For comparison, we shall also study the case where the firms do not interact with their business partners, meaning that contagion effects are not present. In this situation we replace the extremal invariant distribution ν_p of firms' liquidity state in (5) with a product π_ρ of Bernoulli measures with density ρ. For the loss approximation we then obtain

$$\Psi^{\pi}_{d,a}(r) = \int \Phi\left(\frac{\sqrt{r}m(\rho, l_0(k), l_1(k)) - a/\sqrt{r}}{\tilde{\sigma}(k,\rho)} \right) \kappa(\mathrm{d}k), \tag{21}$$

with limiting variance $\tilde{\sigma}^2(k,\rho)$ given by

$$\tilde{\sigma}^2(k,\rho) = (1-\rho)\cdot \mathrm{var}(M_{k,0}) + \rho \cdot \mathrm{var}(M_{k,1}) + \rho(1-\rho)\cdot (l_1(k) - l_0(k))^2,$$

where, in the current context, $\mathrm{var}(M_{k,x}) = P_x(k)(1 - P_x(k))$.

We now consider a portfolio of size $r = 10\,000$, where the probability ρ of an individual firm to be liquidity stressed is equal to 50%. As summarized in Table 1, for fixed complexity parameter d the strength of the contagion effect will be governed by the properties of the distribution of m which depends on the macro-economic factor K. The variable m describes the average loss per firm in the whole economy which depends on the macro-economic state of the economy according to the Bernoulli mixture specification (18). We investigate low and large variance of the macro-factor which is an important determinant for the relevant properties of the distribution of m. We also compare different means of K.

In Figs. 5–7 we plot the approximate loss distribution for different specifications of the distribution of the macro-factor K. That is, we assume K to be distributed according to

(d) a Dirac measure placing mass one on the value 0 (Fig. 5);
(e) a Gaussian distribution with mean 0 and variance 1 (Fig. 6); and
(f) a Gaussian distribution with mean −4 and variance 0.05 (Fig. 7).

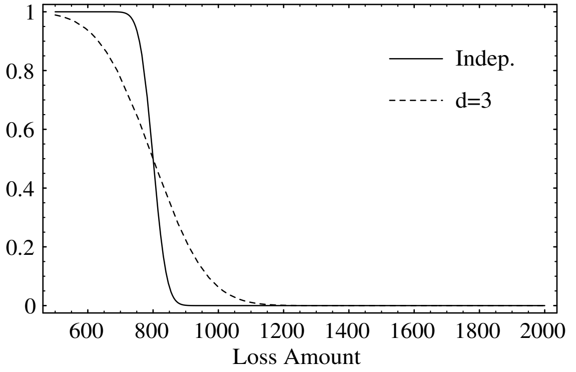

Fig. 5. Approximate loss distribution in the one-factor Probit normal model (19) when the macro-parameter K is certain and set equal to zero, for $d = 3$ and the independence case ($r = 10\,000$ and $\rho = 0.5$).

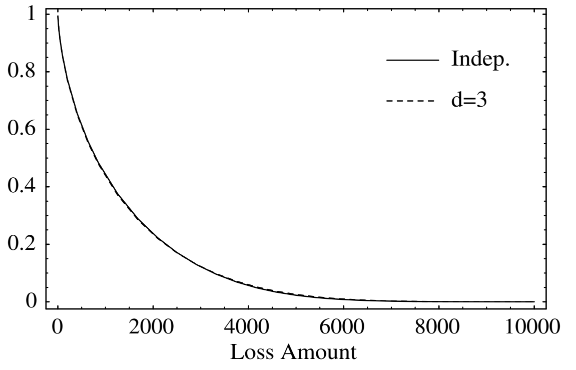

Fig. 6. Approximate loss distribution in the one-factor Probit normal model (19) when the macro-parameter K is standard normally distributed, for $d = 3$ and the independence case ($r = 10\,000$ and $\rho = 0.5$).

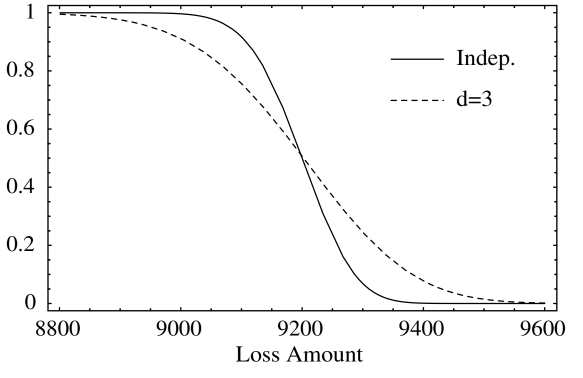

Fig. 7. Approximate loss distribution in the one-factor Probit normal model (19) when the macro-parameter K is normally distributed with mean −4 and variance 0.05, for $d = 3$ and the independence case ($r = 10\,000$ and $\rho = 0.5$).

The qualitative results derived below are robust with respect to small perturbations of the parameters; specific choices are necessary for the numerical calculations, though.

In the cases (d) and (f) the variance of K is very small. This means that there is not a high degree of uncertainty about the state of the macro-economy. From (19), the difference between (d) and (f) is, however, that the latter economy is in worse state (positive values of K describe an expanding economy, while negative ones are describing a downturn). In case (e) the economy is on average between expansion and recession as in (d), but with a high degree of uncertainty (high fluctuation possible).

In the figures we compare the case where firms are independent (corresponding to (21)), and where firms do interact in an economy with degree of complexity $d = 3$

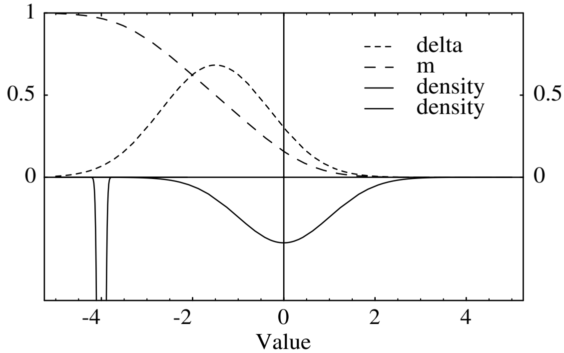

Fig. 8. Upper half: Expected loss difference $\Delta(k)$ and average loss in the whole economy $m(k)$ as functions of the macro-parameter k. Lower half: Densities of Gaussian variables with parameters $(-4, 0.05)$ and $(0, 1)$.

(corresponding to (12) and (20)). In contrast to Figs. 2–4 we do not plot densities, but excess probabilities; thus, the size of the contagion effect is measured by the difference between the two functions in Figs. 5–7, respectively.

In cases (d) and (f) a considerable contagion effect is present with higher probabilities for large losses for contagion-prone than for independent firms (Figs. 5 and 7). In case (e) the difference between the loss probabilities has almost disappeared and no significant contagion effect is visible (Fig. 6).

The different size of the contagion effects can easily be understood if we recall our discussion from Section 3.3 (see Table 1). The contagion effect is governed by the marginal distribution of average losses m, which can be derived from Fig. 8. In the upper half of Fig. 8 the variables m and Δ are shown as a function of k for the one-factor Probit normal model (19) and the given parameter values. The lower half displays the densities of the Gaussian distributions of factor K in the case (e) and (f), respectively.

Due to the low variance of the Gaussian factor distribution in case (f) K takes values close to its expectation with very high probability. Close to the expected value -4 of the factor K, the slope of $k \mapsto m(k)$ is not very large. Hence, the distribution of m is close to a Dirac measure. Consistent with our discussion in Section 3.3 we observe in Fig. 7 a significant contagion effect. Fig. 8 shows that for low values of k the variable m takes values close to one corresponding to a bad macro-environment. Conversely, in case (d) K is deterministic and equal to 0 giving rise to a low deterministic value for m which corresponds to a good macro-economic environment. As expected, comparison of Figs. 7 and 5 shows considerably higher losses in case (f) than in case (d).

Let us now investigate the cases (d) and (e) in which the macro-economic factor K has the same expectation 0 but different variance. If K is deterministic, m is deterministic. By our discussion from Section 3.3 we expect a visible contagion effect as confirmed by Fig. 5. In contrast, if the distribution of K is a centered Gaussian with large variance, Fig. 8 shows that the distribution of m is atomless placing considerable mass on an interval of significant size. This property of the distribution of m corresponds to a small contagion effect (Fig. 6) as we have already noticed in the case of a uniform distribution of m in our discussion in Section 3.3 (see also Fig. 4).

From an economic point of view the bottom line is as follows. The higher the uncertainty about the state of the economy K, the less influence has contagion on the loss distribution. In cases (d) and (f) where the macro-state is (almost) certain, contagion processes lead to considerable fluctuations of losses around their means. In case (e) where the macro-state is quite uncertain and may itself be quite fluctuating, contagion has almost no effects on the loss distribution, which in this case is dominated by the macro-uncertainty. In other words, cyclical effects are of *first order*, while contagion is of *second order* with respect to the fluctuations of losses. This is consistent with the empirical findings, in particular in the bank contagion literature, discussed in Section 1. Nevertheless, the second-order nature of contagion effects does not imply that we should neglect them in the measurement of aggregated credit risks. Contagion does have a quite important effect on the loss distribution, if uncertainty about the macro-economic environment is low. Indeed, if the macro-environment is bad, then contagion processes may cause a significantly higher risk of large losses (see Fig. 7).

5. On estimating the model

In the last section we compared properties of the loss distribution for different model specifications and degrees of connectedness of the economy. In this section we outline the estimation of the model from historical data.

We suppose we are given a sufficiently large set of historical default and loss data. Possible sources include the default and recovery data frequently published by the public credit rating agencies, such as Moody's or Standard & Poor's, as well as data collected internally in financial institutions on proprietary portfolio positions. In a first step we discriminate the entities in the data in "liquidity-stressed" and "liquidity-stable" firms, which correspond to the states $x \in \{0,1\}$. For this we can use, for example, the external or internal credit rating of a firm, or balance-sheet and cash flow data if available.

The next step consists of choosing the conditional distribution $M_{k,x}$ of position losses on a firm in liquidity state $x \in \{0,1\}$ when the state of the economy equals $k \in \mathbb{R}^p$. Because of its practical relevance, we will consider the generalized Bernoulli mixture specification (18), i.e. we put

$$M_{k,x} = \begin{cases} 1 & \text{with probability } P_x(k), \\ 0 & \text{with probability } 1 - P_x(k). \end{cases} \tag{22}$$

$P_x(k)$ can be interpreted as the probability of default for a firm in liquidity state x when the economy is in state k. Following a parametric estimation strategy, we will fix some parametric model for $P_x(k)$ together with a distribution for the macro-factors k. Standard industry-examples as discussed in Section 4 include the Gamma model, the Probit model, and the Logit model, together with the appropriate factor distribution κ.

Note that in contrast to standard models which neglect contagion, we separated the data into two pools depending on the liquidity state of the individual firms. In particular, from this the empirical distribution of the default probabilities conditional on the state $x \in \{0,1\}$ of the firms can be obtained. Hence, we can estimate the parametric models $P_0(K)$ and $P_1(K)$, respectively. In contrast to standard industry practice, in a contagion-based approach $P_0(K)$ and $P_1(K)$ must be estimated under the restriction that in both cases the *same* parameters are chosen.

In a next step, we estimate the distribution Q of the average number of liquidity-stressed firms ρ. For each point in time, the average number of liquidity-stressed firms can be calculated from the data, allowing us to estimate Q.

Taking the parameter d as given, we are now in a position to calculate approximate loss distributions for large portfolios from (12). With the generalized Bernoulli specification, we have in fact that the approximate probability of aggregate losses exceeding $a > 0$ for a portfolio of size $r \in \mathbb{R}_+$ is given by

$$\Psi_{d,a}(r) = \int\int \Phi\left(\frac{\sqrt{r}(\rho P_1(k) + (1-\rho)P_0(k)) - a/\sqrt{r}}{(P_1(k) - P_0(k))\sigma(d,\rho)r^{1/d}} \right) Q(\mathrm{d}\rho)\kappa(\mathrm{d}k), \tag{23}$$

where $\sigma(d,\rho)$ is given by (A.1). Heuristically, by interpolation between various degrees of interaction we may and will actually assume that d is not necessarily a natural number, but can take on any real value larger than 2.

The parameter d stands for the degree of complexity of the business partner network; as discussed in Section 3 it governs the size of the contagion effect present in the economy. Given a homogeneous portfolio of firms, we need to determine its degree of connectedness d if we wish to calculate its loss distribution. To do so, we introduce contagion indicators and contagion rating classes $C = \{c_1, \dots, c_m\}$. Contagion indicators can for example be the number of business partner relations or the number of trade credit relationships an individual firm possesses on average. We will assume that we can assign a contagion rating to a homogeneous portfolio via the indicators.

With every contagion rating class $c \in C$ we will associate a contagion parameter d using our historical data. For a contagion class $c \in C$, the historical loss distribution can be estimated. Comparison with the loss distribution generated by the model for various degrees of connectedness d allows us to estimate the contagion parameter d related to any rating class in C. We emphasize that in the choice of d we can allow for real numbers larger than 2.

Finally, suppose we have estimated the model and we are interested in predicting the loss distribution for some given actual credit portfolio. The contagion indicators can be used to obtain the contagion rating of the portfolio, which in turn corresponds to a contagion parameter d that was obtained by our calibration procedure. By (23), we can now calculate the approximate loss distribution of the portfolio.

6. Conclusion

A thorough understanding of aggregate credit loss risk associated with large portfolios of financial positions is of critical importance for the management of financial institutions and the regulatory authorities supervising financial markets. In aggregating individual risk exposures the dependence between losses on positions is a significant factor. In that respect the standard Bernoulli mixture models widely applied in the financial industry focus exclusively on cyclical correlations between firms' positions, which are due to the dependence of firms on the common macro-environment. Because of its ignorance of default contagion processes such an approach might underestimate aggregate loss risks. In response to that, in this paper we model the local interaction of firms with their business partners and the associated contagion processes, in addition to cyclical correlation effects. We explicitly approximate the distribution of aggregate credit losses on large financial portfolios and investigate the relative strength of cyclical correlations and contagion. When macro-economic quantities are not volatile, the approximate loss distribution is Gaussian with variance of larger order in the number of positions than in the case of independent firms. If the macro-economic factors are highly uncertain, the resulting loss distribution will typically not be normal and may possess fat tails.

With the loss distribution at hand we are able to quantify the relation between the variability of global (macro-) economic fundamentals, strength of local interaction between firms, and the fluctuation of portfolio losses, i.e. the degree of aggregated credit loss risk. If the volatility of macro-economic factors is large, portfolio loss distributions are mainly governed by the distribution of fundamentals. When the main source of risk is reduced and macro-economic uncertainty is small, contagion can have a significant effect on portfolio losses by increasing the probability of large losses. [10] As recently confirmed by empirical studies, the strength of the contagion-induced loss variability and hence the probability of large losses depends on the degree of complexity of the business partner network, i.e. the degree of connectedness between firms in the economy. The more complex the economy and the denser the business partner network, the lower is the contagion-induced additional risk of large losses.

For regulatory authorities our results have the following implications. First, credit contagion phenomena cause additional loss risks, which are not identified by the standard industry models. In particular, conditional on a given macro-economic scenario, this additional risk can be significant. The potential underestimation of total credit loss risk can lead to capital provisions which may prove to be insufficient to buffer actual losses. We identified when such issues become important. Second, the effects of credit contagion are less severe in an economy in which firms operate within a complexly structured business partner network. Regulatory policy supporting complexity

[10] Nevertheless, contagion effects are broadly modest if compared to systematic risk. The contagion effect is only *relatively* large, when the main source of risk, namely the systematic uncertainty, is small. We are grateful to an anonymous referee for pointing this out.

and diversity in business relations among firms thus helps to mitigate adverse credit contagion effects and reduce the degree of systemic risk in the financial market.

Acknowledgements

We are grateful for comments of an anonymous referee. We acknowledge financial support by Deutsche Forschungsgemeinschaft.

Appendix A. Normal approximation

The constant $\sigma^2 = \sigma^2(d, \rho)$ is given by

$$\sigma^2 = \rho(1-\rho)\cdot\frac{\gamma_d\cdot d}{2^{d+3}\pi^{d/2}}\cdot\Gamma\left(\frac{d-2}{2}\right)\cdot\int_{[-1,1]^d}\int_{[-1,1]^d}\frac{1}{\|x-y\|_2^{d-2}}\,\mathrm{d}x\,\mathrm{d}y, \tag{A.1}$$

where Γ is the Gamma-function and $\gamma = \gamma_d$ is given by

$$\frac{1}{\gamma} = (2\pi)^{-d}\int_{(-\pi,\pi)^d}\left(1-\frac{1}{d}\sum_{m=1}^{d}\cos x_m\right)^{-1}\mathrm{d}x.$$

Numerical values of γ_d can be found in Kondo and Hara (1987) for various d.

Proof of Theorem 3.1. Assume first that $Q = \delta_p$ and $\kappa = \delta_k$ for $\rho \in (0, 1)$ and $k \in \mathbb{R}_+$. In this case, the approximation reduces to the case of Theorem 4.4. of Giesecke and Weber (2002), and we get that

$$\sup_{a\in\mathbb{R}}\left|\beta(L_n \geqslant a) - \Psi_{d,a}(|\Lambda_n|;\rho,\tilde{l}(k))\right| \leqslant \epsilon_n, \tag{A.2}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

For given k and ρ, the distribution of

$$|\Lambda_n|^{-\frac{d+2}{2d}}(L_n - |\Lambda_n|\cdot m(\rho,k))$$

under the measure

$$\int(\otimes_{i\in\mathbb{Z}^d}M_{k,\xi(i)})(\mathrm{d}w)v_\rho(\mathrm{d}\xi), \quad w\in\mathbb{R}_+^{\mathbb{Z}^d},$$

will be denoted by $\varsigma^n_{\rho,k}$. We define the quantity

$$\delta^n_{\rho,k} := \sup_{n'\geqslant n}\sup_{z\in\mathbb{R}}\left|\varsigma^{n'}_{\rho,k}([z,\infty)) - \Phi\left(-\frac{z}{(l_1(k)-l_0(k))\cdot\sigma(\rho)}\right)\right|,$$

where Φ is the Gaussian distribution function.

Inequality (A.2) implies that $\delta^n_{\rho,k}$ converges to 0 for all $\rho \in (0, 1)$ and $k \in \mathbb{R}^p$ as $n \to \infty$. Observe that $(\rho, k) \mapsto \delta^n_{\rho,k}$ is measurable. For $\epsilon > 0$ we can therefore define measurable sets

$$A^n_\epsilon = \{(\rho, n) \in (0,1) \times \mathbb{R}^p : \delta^n_{\rho,k} < \epsilon\}.$$

Then $A^n_\epsilon \subseteq A^{n+1}_\epsilon$, and $(Q \otimes \kappa)(A^n_\epsilon) \nearrow 1$ as $n \to \infty$. Choose n_0 large enough such that

$$(Q \otimes \kappa)(A^{n_0}_\epsilon) \geqslant 1 - \epsilon.$$

Let $(\rho, k) \mapsto z(\rho, k)$ be a measurable mapping. Then for all $n \geqslant n_0$ we get

$$\begin{aligned} &\left| \int \left[\varsigma^n_{\rho,k}([z(\rho,k),\infty)) - \Phi\left(\frac{z(\rho,k)}{(l_1(k) - l_0(k))\sigma(\rho)} \right) \right] Q(\mathrm{d}\rho)\kappa(\mathrm{d}k) \right| \\ &\quad \leqslant 2(1 - (Q \otimes \kappa)(A^n_\epsilon)) + \sup_{(\rho,k)\in A^n_\epsilon} \sup_{z'\in\mathbb{R}} \left| \varsigma^n_{\rho,k}([z',\infty)) - \Phi\left(\frac{z'}{(l_1(k) - l_0(k))\sigma(\rho)} \right) \right| \\ &\quad \leqslant 3\epsilon. \end{aligned}$$

Let $a \in \mathbb{R}$ be arbitrary, and let $n \geqslant n_0$. We can choose

$$z(\rho, k) = |\Lambda_n|^{-\frac{d+2}{2d}}(a - |\Lambda_n| m(\rho, k)).$$

It follows that for any $a \in \mathbb{R}$ and $n \geqslant n_0$ the following inequality holds:

$$\begin{aligned} &\left| \int \int v_\rho(L_n \geqslant a) Q(\mathrm{d}\rho)\kappa(\mathrm{d}k) \right. \\ &\quad \left. - \int \int \Phi\left(\frac{|\Lambda_n|^{1/2} m(\rho,k) - |\Lambda_n|^{-1/2} a}{\sigma(\rho)|\Lambda_n|^{1/d}} \right) Q(\mathrm{d}\rho)\kappa(\mathrm{d}k) \right| \leqslant 3\epsilon. \end{aligned}$$

This completes the proof. □

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