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Credit contagion and aggregate losses

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Abstract

Credit contagion refers to the propagation of economic distress from one firm to another. This article proposes a reduced-form model for these contagion phenomena, assuming they are due to the local interaction of firms in a business partner network. We study aggregate credit losses on large portfolios of financial positions contracted with firms subject to credit contagion. In particular, we provide an explicit Gaussian approximation of the distribution of portfolio losses. This enables us to quantify the relation between the volatility of losses and the determinants of credit contagion. We find that contagion processes have typically a second-order effect on portfolio losses. They induce additional fluctuations of losses around their averages, whose size depends on the number of business partners of the firms.

JEL classification: G11; G31

Keywords: Credit contagion; Business partner network; Credit portfolio loss distribution; Portfolio loss volatility; Voter model

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1. Introduction

Defaults of firms are stochastically dependent. One reason is that firms' financial health is sensitive to macro-economic factors, such as energy prices, GDP growth, or interest rates. These factors are common to all firms operating in an economy. The fluctuation of factors affects firms simultaneously and induces *cyclical default dependence*. Another reason for default dependence is the existence of business ties between firms. These links often provide the channel for the propagation of economic distress from one firm to another. This is called *credit contagion*. In this paper we propose a model for such contagion phenomena and study the contagion-induced volatility of aggregate credit losses on large portfolios of financial positions. The measurement of aggregated risk is essential for the management and regulation of financial institutions.

Borrowing and lending networks constitute a typical distress propagation channel. In the banking sector, *interbank lending* refers to banks' mutual claims. To the extent that interbank loans are neither collateralized nor insured against, the distress of a bank may trigger the subsequent distress of other banks in the lending chain. Allen and Gale (2000) propose an equilibrium model for such phenomena where different sectors of the banking system have overlapping claims on one another in order to buffer liquidity preference shocks. This arrangement is however financially fragile: depending on the degree of connectedness of the buffer system, a small liquidity preference shock in one bank can spread through the economy and cause the distress of other banks as well. In the manufacturing sector, *trade credits* link suppliers and buyers of goods through a chain of obligations. Kiyotaki and Moore (1997) study how a liquidity shock, which causes the distress of an individual customer in the first place, can propagate through the borrowing–lending network and result in a chain reaction.

While insightful, these micro-economic models cannot quantify aggregated loss risk due to contagion. In order to derive explicitly the distribution of aggregated losses due to contagion, we propose a reduced-form contagion model. Our approach adopts the micro-economic reasoning of Allen and Gale (2000) and Kiyotaki and Moore (1997), but models local firm interaction statistically.

We consider a homogeneous economy that hosts a large number of firms that share the same individual characteristics. The business partner network is represented by a d-dimensional lattice. The nodes are identified with firms. The edges represent business partner relationships.

The financial health of a firm is characterized by the amount of available liquidity. We specify two states, "high liquidity" and "low liquidity," the latter describing a firm that is financially distressed. The initial state of a firm is random. Over time, a firm migrates between states, reflecting a dynamic business environment. A state transition is a Poisson event, whose intensity depends on the state of the firm's business partners. A firm's transition intensity is proportional to the number of the firm's business partners that are in the opposite state. The intuition is that a financially distressed firm is likely to default on payment obligations. The more distressed partners a healthy firm has, the higher the likelihood that the firm suffers a liquidity shortage and becomes distressed as well.

The joint evolution of firms' states over time is described by a continuous-time Markov process. This process is also known as the *voter model* in the theory of interacting particle systems (Liggett, 1985). We analyze the asymptotic behavior of the liquidity process. The structure of the invariant (equilibrium) distribution of liquidity states depends on the dimension d of the lattice that represents the firm network. If the dimension is low ($d = 1, 2$), firms have only a few business partners. Then an individual firm is highly dependent on each of these partners. Clusters of firms in the same state are relatively stable. Their size fluctuates randomly; they grow and merge with other clusters. In the long run, all firms are in the same state. This implies a high degree of systemic risk. In their micro-economic model, Allen and Gale (2000) obtain a qualitatively similar behavior: with a simple lending network structure, firms are financially fragile and the degree of systemic risk is high.

If the dimension of the lattice is high ($d > 2$), then firms have many business partners and the equilibrium distribution of liquidity states becomes non-trivial. Random clusters of firms in the same state appear only locally and their size fluctuates. In particular, they do not merge and grow in the same way as in the low-dimensional case, but they are more unstable and less persistent. There are again qualitative parallels to the micro-economic contagion models of Allen and Gale (2000) and Kiyotaki and Moore (1997). In a borrowing/lending network in which firms have many business partners, firms are more robust with respect to liquidity shocks, which are buffered through the network.

We investigate the structure of the equilibrium liquidity distribution in a large homogeneous economy, where firms are equal with respect to their marginal liquidity risk. In the ergodic case, the equilibrium liquidity states are governed by a so-called extremal distribution corresponding to the fixed degree of marginal liquidity risk. In the general case, the equilibrium liquidity states are governed by a mixture of extremal distributions. The mixing distribution corresponds to the distribution of the average number of low-liquidity firms in the whole economy, which is a random quantity. It can hence be thought of as describing systematic risk. The mixing distribution, as well as the expected proportion of low-liquidity firms, is not changed through the interaction of firms. What interaction changes is, however, the dependence between firms' states. For any finite number of firms, the probability to find many firms in the same state is higher than with independent firms.

For a fixed horizon, we are interested in the distribution of aggregate losses that a financial institution suffers from positions contracted with firms subject to credit contagion. We assume that the loss on a position with a given firm is random and depends on the firm's liquidity state. Given the firms' states, losses are independent. We base our assessment of aggregate loss risk on the equilibrium liquidity distribution in a network of firms with many business partners.

Average losses on infinitely large portfolios are governed by the average proportion of low-liquidity firms and the expected conditional position losses. While loss uncertainty stemming from the fluctuation of position losses averages out, loss uncertainty remains from the average proportion of low-liquidity firms. The randomness in average losses is hence governed by the mixing distribution, which represents systematic risk. Contagion effects play no role in infinitely large portfolios; they are diversified away entirely.[1] This is confirmed by an analysis of the quantiles of the loss distribution. In the large portfolio limit, these are basically governed by the quantiles of the mixing distribution.

Losses in finite portfolios are our main concern. We provide an *explicit* Gaussian approximation to the distribution of losses in finite portfolios. This approximation is based on a recent central limit theorem for the general voter model proved by Zähle (2001), which is non-classical due to the strong dependence induced by the local interaction. The approximation is the key to the measurement and management of the portfolio's aggregated risk.

We analyze the determinants of the volatility of losses in finite portfolios. As in infinite portfolios, average losses are random and governed by the distribution of systematic risk. However, in finite portfolios contagion induces a second-order effect on the volatility of losses. It causes additional fluctuations of losses around their (random) averages, so that the probability of large losses is elevated through contagion. The amount of additional loss volatility depends on two quantities: the characteristics of the systematic risk in the economy and the dimension d of the lattice representing the firm network. Through numerical calculations we illustrate that the effect of contagion on losses decreases with increasing volatility of systematic risk and increasing d.

Our approximation results complement the existing literature on large homogeneous credit portfolios, which neglects credit contagion and instead focuses on cyclical default dependence: Vasicek (1987), Frey and McNeil (2002, 2003), Lucas et al. (2001), Schloegl (2002) and Gordy (2003). In these models, the volatility of aggregate losses is entirely due to the fluctuation of some exogenous macroeconomic variables. Based on the results in this paper, Giesecke and Weber (2004b) provide an explicit approximation that integrates cyclical and contagion effects. This allows us to quantify the relation between the volatility of losses, volatility of macro-factors and the dimension of the lattice. Alternative approaches to modeling cyclical and contagion effects include those of Frey and Backhaus (2003), Giesecke (2004), Giesecke and Goldberg (2004), Jarrow and Yu (2001) and Schönbucher and Schubert (2001). These contributions focus on the pricing of multiname credit derivatives. They do not derive the loss distribution analytically. A model that focuses on contagion effects as we do is proposed by Davis and Lo (2001). They explicitly derive an analytic expression for the exact distribution of portfolio losses.

The paper is organized as follows. In Section 2 we propose a statistical model for credit contagion. In Section 3, we analyze the asymptotic behavior of the liquidity process and the structure of the equilibrium liquidity distribution. In Section 4 we provide an explicit approximation of the distribution of aggregate losses. Section 5 concludes by discussing the model assumptions. All proofs are in the appendix.

[1] Horst (2004) considers a model where contagion effects do not diversify away in the limit.

2. Modeling credit contagion

We provide a statistical model for the effects of credit contagion and investigate their consequences on the level of both the whole economy and large portfolios.

2.1. A reduced-form model

We consider an economy with a collection S of small or medium-sized firms which is at most countably infinite. A firm $i \in S$ can be in two liquidity states, denoted 0 and 1. State 0 is interpreted as "high liquidity," while state 1 is interpreted as "low liquidity." The state of the economy is characterized by a configuration in the state space $\{0,1\}^S$. We are interested in the evolution of firms' liquidity states over time and the interdependence of the states of different firms.

The liquidity state of an arbitrary firm i is influenced by the state of a collection $N(i) \subseteq S \backslash \{i\}$ of business partners. We assume that any firm $i \in S$ is a creditor of its business partners. At a time τ some business partner $j \in N(i)$ is obliged to pay a certain amount to its creditor i. Depending on its liquidity state at the maturity date τ, firm j will or will not fulfill its obligation. We suppose that firm j pays its debt if it is in state 0. If it is in state 1, then it defaults on its obligation. Hence the state of obligor j influences the liquidity state of the creditor firm i. If j fulfills its obligation, creditor i is in state 0 from time τ onwards. Otherwise, i is in state 1 after time τ.

In a large economy, modeling explicitly all borrowing and lending relationships becomes extremely complex and is not tractable. To reduce the complexity of the problem, we provide a statistical model for the interaction of the firms. In contrast to standard micro-economic models, we thus describe the choice of the obligor $j \in N(i)$ and the maturity date τ in a probabilistic way. We assume that τ is a random time that is standard exponentially distributed. The business partner of firm i whose payment is due at time τ is chosen according to some distribution $p(i,j)$ where $j \in N(i)$.

We denote the liquidity state of a firm $i \in S$ by $\xi(i)$. Given our assumptions, the transition between liquidity states is a Poisson event. The transition rate of the state of firm $i \in S$ can formally be written as

$$c(i,\xi) = \sum_{j: j \in N(i)} p(i,j)|\xi(i) - \xi(j)|.$$

In this sense our credit contagion model belongs to the class of *reduced-form* credit risk models (see, e.g., Jarrow and Turnbull (1995), Duffie et al. (1996), Duffie and Singleton (1999), Jarrow et al. (1997) and Lando (1998) for single-firm models). The idea that a firm's default intensity directly depends on the state of its counterparties appears also in Jarrow and Yu (2001) and Davis and Lo (2001).

2.2. The voter model

The evolution of the liquidity state of an arbitrary firm i is influenced by the state of a collection $N(i) \subseteq S \backslash \{i\}$ of business partners. $N(i)$ will be called the set of *neighbors* of firm i. For simplicity, we assume that firms influence each other in a *symmetric* way: if firm i's state is influenced by firm j, then firm j's state is influenced by firm i. Expressed in terms of the neighborhoods,

$$j \in N(i) \quad \Rightarrow \quad i \in N(j).$$

If we connect all firms $i \in S$ to their neighbors $j \in N(i)$, we get an undirected graph which characterizes the business relations of the firms. Business partners are nearest neighbors on the graph. For tractability, we assume a simple neighborhood structure which is specified by a d-dimensional lattice. In particular, all firms have the same finite number of business partners. Hence, we consider an economy with a countably infinite number of firms. Firms are identified with their location on the d-dimensional integer lattice $S = \mathbb{Z}^d$. According to our assumptions, at a unit exponential time the payment of a business partner j of firm i is due. Firm j is chosen according to some distribution $p(i,j)$ where $|j - i| = 1$. Here $|\cdot|$ denotes the length of the shortest path between two firms on the lattice. To keep the analysis simple, we choose $p(i,j)$ to be the uniform distribution, i.e. $p(i,j) = 1/2d$. The contagion pattern we proposed above implies that the transition rate c is given by

$$c(i,\xi) = \frac{1}{2d} \sum_{j:|i-j|=1} |\xi(i) - \xi(j)| = \begin{cases} \dfrac{1}{2d} \displaystyle\sum_{j:|i-j|=1} \xi(j) & \text{if} \quad \xi(i) = 0, \\ \dfrac{1}{2d} \displaystyle\sum_{j:|i-j|=1} [1 - \xi(j)] & \text{if} \quad \xi(i) = 1. \end{cases}$$

The transition rate c is a function of the firm $i \in \mathbb{Z}^d$ and the liquidity configuration $\xi \in X := \{0,1\}^{\mathbb{Z}^d}$. A regular version of the process is given by the *voter model*. The voter model is well known in the theory of interacting particle systems (Liggett, 1985, 1999). The evolution of firms' liquidity states is described by a continuous-time Feller process $(\eta_t)_{t \geqslant 0}$ with state space X and transition rate c. Here $\eta_t(i)$ is the liquidity state of firm i at time t.

The rate at which firm i switches its state is represented by $c(i,\xi)$. That is, a firm $i \in \mathbb{Z}^d$ with a high-liquidity state ($\xi(i) = 0$) migrates to a low-liquidity state ($\xi(i) = 1$) at a rate proportional to the number of low-liquidity neighboring firms $j \in \{j: \xi(j) = 1, |i - j| = 1\}$, and vice versa. Put another way, after a unit exponential waiting time in one state, a firm $i \in \mathbb{Z}^d$ migrates to the state of some neighboring firm $j \in \{j: |i - j| = 1\}$ which is chosen with probability $1/2d$. A transition is hence a Poisson event, whose intensity is proportional to the number of neighboring firms with opposite liquidity state.[2] It is easy to see that if all firms $i \in \mathbb{Z}^d$ are either in good or in bad shape, then the transition rate is zero.

This formal model of the joint evolution of firms' liquidity states probabilistically describes the pattern of credit contagion phenomena as we introduced them above.

[2] We could multiply the transition rate c of the voter model by an arbitrary constant without changing the long-run behavior of the dynamics. The modified rate translates into a deterministic linear time change of the model. As pointed out by a referee, it would be very interesting to investigate the speed of convergence to equilibrium.

Pick the specific example of trade credit. If some business partners of a high-liquidity firm in a trade credit are in the low-liquidity state, then the probability that this firm migrates to the low-liquidity state due to a payment default in the credit chain increases with the number of low-liquidity partners. If some business partners of a low-liquidity firm are in the high-liquidity state, then the probability of that firm's migration to the high-liquidity state increases with the number of healthy partners.

3. Equilibrium behavior

We consider the equilibrium distributions[3] and the asymptotic behavior of the liquidity process η. It turns out that the structure of the equilibrium distributions depends on the dimension d of the lattice representing the firm network. The larger the d, the more business partners any individual firm has. At the same time the number of indirect inter-firm links of given length increases.[4]

3.1. Networks of firms with few business partners

The liquidity state of an obligor i is revealed to its creditor $j \in N(i)$ at the maturity of an obligation. Firm i will default if and only if it is in the low-liquidity state. Conversely, immediately after maturity, the liquidity state of both firms i and j will be equal according to the contagion process. Thus, also the subsequent state of firm j will be known to firm i. Hence, for firms *inside* the network the liquidity states of business partners are partially observable. For financial institutions *outside* of the network of small or medium-sized firms, we suppose in contrast that the liquidity state of firms $i \in \mathbb{Z}^d$ cannot be observed. Hence the liquidity configuration of the firms is random and described by a probability distribution on the state space X.

At some initial time the distribution of η is given by the distribution μ on X. We are interested in the behavior of the liquidity process in the long run. The process η has càdlàg paths; for convenience, we will work with the canonical version of the process. Ω denotes the space of càdlàg functions on $\mathbb{R}_+$ with values in X endowed with the usual augmented filtration. For the law of the process η with initial distribution μ, we will write P^μ.

We shall assume that μ is translation invariant[5] and denote by

$$\rho = \mu\{\xi\colon \xi(i) = 1\} \tag{1}$$

[3]An *equilibrium distribution* is a probability measure on the state space X that is invariant under the Markovian dynamics of the voter model.

[4]More specifically, if i and j are two firms, a sequence $(i_0, i_1, \dots, i_n)$ of firms is a link of length n between i and j, if i_k is a neighbor of i_{k+1} for $k \in \{0, 1, \dots, n-1\}$, $i_0 = i$ and $i_n = j$. The number of links of length n emanating from a given firm equals $2d(2d-1)^{n-1}$ and grows exponentially in n and polynomially in d.

[5]*Translation invariance* generalizes the notion of *stationarity* for stochastic processes to the multidimensional case. For $\xi \in X$ and $i \in \mathbb{Z}^d$ we define the translation $T_i(\xi)(j) = \xi(i+j)$. Canonically, the translation T_i operates also on subsets of X. A measure μ on X is called translation invariant, if $\mu(A) = \mu(T_i A)$ for all $i \in \mathbb{Z}^d$ and for all measurable $A \subseteq X$.

the Bernoulli parameter of the initial marginal liquidity distribution for an arbitrary firm i. ρ can hence be thought of as a measure of an individual firm's marginal liquidity risk. In particular, the translation invariance of μ implies that the firms in the economy are *homogeneous* with respect to marginal risk.

For $d = 1, 2$ and translation-invariant initial law μ, as $t \to \infty$ the distribution of η_t converges weakly to the mixture

$$\rho\delta_1 + (1-\rho)\delta_0, \tag{2}$$

cf. Liggett (1999). Here δ_ξ is the Dirac measure placing mass 1 on configuration $\xi \in X$. In (2) the indices 0 (1) refer to the configurations with *all* firms being in high (low)-liquidity state. The liquidity process η *clusters*, i.e. for all $i, j \in \mathbb{Z}^d$

$$\lim_{t\to\infty} P^\mu[\eta_t(i) \neq \eta_t(j)] = 0. \tag{3}$$

If firms have only a few business partners, then in the long run only one firm type appears: with probability ρ *all* firms are in the low-liquidity state, and with probability $1-\rho$ *all* firms are in the high-liquidity state. The marginal liquidity distribution of any individual firm is invariant under the contagion dynamics: the degree of marginal risk is not affected by the interaction process. Nevertheless, the economy can change drastically on the macroscopic level.

This behavior is quite intuitive in the trade credit chain interpretation. If initially the marginal probability ρ of individual firms to be in the low-liquidity state is high, then it is quite likely that high-liquidity firms in the credit chain become "infected." Random clusters of firms in the low-liquidity state emerge with high probability, while clusters of firms in the high-liquidity state emerge only with low probability. In any case, if the chain a firm operates in is "short," then the state of the relatively few business partners highly dominates the state of a firm in the chain. Here clusters of firms of the same type are relatively stable. The size of the clusters fluctuates randomly, but for low dimensions $d \leqslant 2$ some of the clusters merge and form large growing clusters. Asymptotically, with high probability ρ *all* firms are in the low-liquidity state, and with low probability $1-\rho$ *all* firms are in the high-liquidity state. Vice versa, if ρ is low, then it is unlikely that a firm gets distressed. In the limit, with high probability $1-\rho$ *all* firms will have the high-liquidity state, with low probability ρ the low-liquidity state.

3.2. Networks of firms with many business partners

The limiting behavior of η differs for dimensions $d > 2$. We analyze the asymptotic behavior of the liquidity process in two steps. First we focus on the special case of *ergodic*[6] initial distributions. Then we derive the long-run behavior of the process for general translation-invariant initial distributions from this special case.

[6]A translation-invariant distribution μ on X is called *ergodic* if $\mu(A) \in \{0, 1\}$ for any translation-invariant subset $A \subseteq X$. Here, a set $A \subseteq X$ is called translation invariant if $T_i A = A$ for all $i \in \mathbb{Z}^d$. Intuitively, a measure μ is ergodic if macroscopic quantities are deterministic, i.e. have probability 0 or 1. This implies, in particular, that strong laws of large numbers are valid. Finally, if μ and ν are both ergodic and not equal, they are mutually singular—that is, they live on different sets.

Before we characterize the asymptotic behavior of the liquidity process, we need to describe the structure of probability measures which are invariant for the voter model. We endow the space $\mathscr{M}_1(X)$ of probability measures on X with the weak topology. The set of probability measures which are invariant for the voter model is denoted by $\mathscr{I}$. The collection of invariant measures $\mathscr{I}$ is a convex set which is closed in the weak topology. The set of extremal points of $\mathscr{I}$ is denoted by $\mathscr{I}_{\mathrm{ex}}$. Here, a measure $v \in \mathscr{I}$ is called an *extremal point* of $\mathscr{I}$, if v is not a proper convex combination of other elements of $\mathscr{I}$. That is, if $v = \alpha v_1 + (1-\alpha)v_2$ for $v_1, v_2 \in \mathscr{I}$, $\alpha \in (0,1)$, then $v = v_1 = v_2$.

It turns out that for $\rho \in [0,1]$ the set $\mathscr{I}_{\mathrm{ex}}$ contains exactly one element v_ρ with $v_\rho\{\xi\colon \xi(i) = 1\} = \rho$. We can therefore label the extremal invariant measures by the Bernoulli parameter of their one-dimensional marginals and obtain in a natural way a one-parameter family

$$\mathscr{I}_{\mathrm{ex}} = \{v_\rho\colon \rho \in [0,1]\}.$$

For the following result we refer to Liggett (1999).

Theorem 3.1. *For any translation-invariant ergodic initial distribution μ with Bernoulli parameter $\rho = \mu\{\xi\colon \xi(i) = 1\}$, as $t \to \infty$ the distribution of η_t converges weakly to the non-trivial* extremal *invariant measure v_ρ of the voter model in dimension d with parameter $\rho = v_\rho\{\xi\colon \xi(i) = 1\}$.*

In contrast to the case $d \leqslant 2$, in this case the contagion process η *coexists*, referring to the lack of clustering of liquidity states in the long run. The average number of low-liquidity firms in the whole economy is a preserved quantity of the dynamics and equals ρ forever.

We study the equilibrium distribution of liquidity states for a general, i.e. not necessarily ergodic initial distribution κ. As stated in the next theorem, the liquidity process η converges weakly to a mixture of the extremal invariant measures v_ρ ($\rho \in [0,1]$) of the voter model. A sufficient statistic for the asymptotic distribution of the process is given by the empirical proportion of low-liquidity firms in the whole economy. It is a standard result that this quantity exists almost surely for translation-invariant measures on X.

Definition 3.2. For a translation-invariant probability measure μ on X the empirical proportion of low-liquidity firms is a random variable $\bar{\rho}$ which is μ-almost surely defined as $\bar{\rho} := \lim_{n\to\infty} |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \xi(i)$, where

$$\Lambda_n := [-n,n]^d \cap \mathbb{Z}^d. \tag{4}$$

By $\mathscr{M}_{\mathrm{e}}$ we denote the class of ergodic probability measures on X endowed with the weak topology. For any translation-invariant probability measure μ on X, the theorem of Choquet states that there is a probability measure γ on $\mathscr{M}_{\mathrm{e}}$ such that

$$\mu = \int_{\mathscr{M}_{\mathrm{e}}} v\gamma(\mathrm{d}v), \tag{5}$$

so that μ can be represented as a mixture of ergodic measures ν. In Theorem A.2 in the Appendix we state a refined Choquet decomposition which can be used to establish the complete convergence theorem for η in case $d>2$ for general translation invariant initial distributions of liquidity states:

Theorem 3.3. *Let $d>2$ and denote by μ_t^κ the distribution of η_t for given initial distribution κ on X. Let κ be a translation-invariant measure, and let*

$$\kappa = \int_{[0,1]} \left(\int_{\mathscr{M}_\mathrm{e}} \nu \gamma_\rho(\mathrm{d}\nu) \right) Q(\mathrm{d}\rho) \tag{6}$$

be the refined ergodic decomposition of κ, cf. Theorem A.2. *Then we have that*

$$\mu_t^\kappa = \int_{[0,1]} \left(\int_{\mathscr{M}_\mathrm{e}} \mu_t^\nu \gamma_\rho(\mathrm{d}\nu) \right) Q(\mathrm{d}\rho), \tag{7}$$

and, letting the symbol "$\xrightarrow{\mathrm{w}}$" denote weak convergence of probability measures,

$$\mu_t^\kappa \xrightarrow{\mathrm{w}} \int_{[0,1]} \nu_\rho Q(\mathrm{d}\rho), \tag{8}$$

where ν_ρ is the extremal invariant measure of the basic voter model in dimension $d>2$ with parameter $\rho \in [0,1]$. Q is the distribution of the empirical proportion of low-liquidity firms in the whole economy under the measure μ.

Proof. See Appendix A. □

The refined ergodic decomposition (6) describes the initial distribution κ of liquidity states as a two-step random process: first the parameter $\rho \in [0,1]$ is chosen according to the distribution Q, which then prescribes the translation-invariant regime

$$\kappa_\rho := \kappa_{\rho,0} := \int_{\mathscr{M}_\mathrm{e}} \nu \gamma_\rho(\mathrm{d}\nu).$$

The distribution Q governs the mixture of the regimes κ_ρ in the decomposition of the initial distribution.

The evolution of the liquidity distribution is described by (7) and (8). If the initial distribution κ can be decomposed as in (6), then the liquidity distributions μ_t^κ at time t and $\mu_\infty^\kappa = \lim_{t\to\infty} \mu_t^\kappa$ can be decomposed analogously. Theorem 3.3 describes these distributions of liquidity states as two-step random processes: first the parameter $\rho \in [0,1]$ is chosen according to the distribution Q, which then determines the regimes

$$\kappa_{\rho,t} = \begin{cases} \int_{\mathscr{M}_\mathrm{e}} \mu_t^\nu \gamma_\rho(\mathrm{d}\nu) & \text{if} \quad t<\infty, \\ \nu_\rho & \text{if} \quad t=\infty. \end{cases}$$

Asymptotically, the distribution μ_∞^κ of liquidity states is a probability-weighted average of extremal invariant measures ν_ρ of the voter model; this mixture is governed by the distribution Q which is given by the initial law of the average number $\bar{\rho}$ of low-liquidity firms in the economy.

Corollary 3.4. *Under the assumptions of Theorem* 3.3, *we can characterize the behavior of the empirical proportion of low-liquidity firms* $\bar{\rho}$ *as follows*:

(1) $\bar{\rho}$ *is* $\kappa_{\rho,t}$*-almost surely equal to* ρ *for* $\rho \in [0,1]$ *and* $t \in [0,\infty]$.
(2) *For* $t \in [0,\infty]$ *the law of* $\bar{\rho}$ *under* μ_t^{κ} *equals* Q.

Proof. See Appendix A. □

The distribution of the average number of low-liquidity firms in the economy is preserved under the contagion dynamics; it is not changed through the interdependence of firms. What interaction between firms changes is the dependence between the liquidity states of different firms. For any finite number of firms, the probability to find many firms in the same liquidity state is higher than in the case of independent firms.

4. Aggregate losses on large portfolios

We consider a financial institution that holds a portfolio of financial positions issued by firms $i \in \Lambda_n \subseteq \mathbb{Z}^d$, where Λ_n is defined in (4). The parameter $n \in \mathbb{N}$ determines the size of the portfolio Λ_n. The positions are subject to credit risk: whether or not an issuer will be able to honor a financial obligation depends on the issuer's state. We wish to assess the bank's aggregated risk of losses at some fixed horizon. Denoting the losses on positions contracted with firm $i \in \Lambda_n$ by the random variable $U(i)$, we are interested in the distribution of portfolio losses

$$L_n = \sum_{i \in \Lambda_n} U(i). \tag{9}$$

We make the following assumptions. Conditional on the liquidity configuration of the firms, losses are independent. The conditional distribution M_r of losses with respect to a firm in liquidity state $r \in \{0,1\}$ depends only on r. We suppose that losses are supported in a bounded interval on $\mathbb{R}_+$. We take M_r as given and let $l_r = \int w M_r(\mathrm{d}w)$ denote the expected loss caused by a firm in liquidity state r. For high-liquidity firms the probability of (large) losses is small relative to firms in the low-liquidity state. M_1 is more concentrated on large values than M_0. Specifically, we might assume that M_1 stochastically dominates M_0, i.e. for all bounded increasing functions $f\colon \mathbb{R}_+ \to \mathbb{R}$: $\int f\,\mathrm{d}M_1 \geqslant \int f\,\mathrm{d}M_0$. We however only suppose that $l_1 > l_0$.

We consider the case $d > 2$ and assume that the economy is in equilibrium, in the sense that the distribution of firms' liquidity states is invariant.

4.1. Deterministic conditional losses

We start by assuming that credit losses $U(i)$ depend deterministically on the liquidity state of firm i. Specifically, we simply set $M_r = \delta_r$ for $r \in \{0,1\}$. This implies that the institution suffers no loss from positions with high-liquidity firms, and a loss of one unit of account from positions with low-liquidity firms.

Let $\mu = \int_0^1 v_\rho Q(\mathrm{d}\rho)$ be an equilibrium distribution of liquidity states. Here, the measures v_ρ ($\rho \in [0,1]$) are the extremal invariant measures of the voter model, and Q is the distribution of the random empirical proportion of low-liquidity firms in the economy which we denote by $\bar{\rho}$. Consider now the average loss $|\Lambda_n|^{-1}L_n$ in portfolio Λ_n. Since the measure v_ρ ($\rho \in [0,1]$) is ergodic, we obtain by a conditional law of large numbers[7]

$$\lim_{n\to\infty} \frac{L_n}{|\Lambda_n|} = \bar{\rho} \tag{10}$$

μ-almost surely. Even with deterministic conditional loss amounts not all loss uncertainty averages out. There is still uncertainty concerning average portfolio losses governed by the distribution Q. This distribution captures the systematic risk in the economy.

The average portfolio loss is thus not governed by the interaction of the firms, but simply by the distribution Q. This is due to the ergodicity of the extremal invariant measures of the voter model. The Eq. (10) relies only on the validity of a law of large numbers and *not* on the specific structure of the ergodic measures in the decomposition of μ. Whenever the ergodic measures have the correct one-dimensional marginal distribution, Eq. (10) holds.

Let us illustrate this in the benchmark case of conditionally independent firms. If we replace v_ρ by a product measure π_ρ of Bernoulli distributions with parameter ρ and consider a distribution $\hat{\mu} = \int_0^1 \pi_\rho Q(\mathrm{d}\rho)$ of liquidity states, contagion is not present any more. The mixture $\hat{\mu}$ corresponds to an economy in which the liquidity states of individual firms are not interdependent via direct business relations—they are only coupled through systematic risk captured by the distribution Q. In this case, Eq. (10) is still valid $\hat{\mu}$-almost surely.

Contagion does not affect the average per capita loss in the economy, but it increases the risk of large losses in *finite* portfolios. This effect can be quantified by a non-classical limit theorem which we state below.

We start with the special case where $Q = \delta_\rho$ for fixed $\rho \in (0,1)$ and investigate the portfolio losses associated with the extremal invariant distribution v_ρ of the liquidity states. The case of general Q is considered later.

Theorem 4.1. *Let $d > 2$ and $Q = \delta_\rho$ for $\rho \in (0,1)$. Suppose additionally that $M_r = \delta_r$ for $r \in \{0,1\}$. For large portfolios the law of the losses L_n can be approximated by a normal distribution*:

$$|\Lambda_n|^{-(d+2)/2d}(L_n - |\Lambda_n|\rho) = |\Lambda_n|^{-(d+2)/2d} \sum_{i\in\Lambda_n} (\xi(i) - \rho) \xrightarrow{\mathrm{w}} \mathcal{N}(0, \sigma^2), \tag{11}$$

[7] In our case, the conditional law of large numbers is simply a reformulation of Corollary 3.4.

where the limiting variance $\sigma^2 = \sigma^2(d)$ is given in Appendix C and numerical values are as follows:

d	3	4	5	6
$\sigma^2/(\rho(1-\rho))$	0.5939	0.4517	0.3765	0.2187

The loss distribution can uniformly be approximated:

$$\sup_{x\in\mathbb{R}_+}\left|\nu_\rho(L_n \geqslant x) - \Phi\left(\frac{|\Lambda_n|^{1/2}\rho - |\Lambda_n|^{-1/2}x}{\sigma|\Lambda_n|^{1/d}}\right)\right| \leqslant \varepsilon_n, \tag{12}$$

where $\varepsilon_n \to 0$ as $n \to \infty$, and Φ is the standard normal distribution function.

Proof. See Appendix B. □

The re-scaling in (11) is non-classical. This is caused by the strong dependence in the equilibrium distribution of liquidity states, which results from the contagion dynamics. Unfortunately, we are not able to provide bounds of Berry–Esseen type for the errors ε_n in (12), which would help to understand the speed of convergence.

By inequality (12) the probability of a loss larger than $x \in \mathbb{R}_+$ can uniformly be approximated by the function

$$\Psi_{d,\rho}(|\Lambda_n|, x) = \Phi\left(\frac{|\Lambda_n|^{1/2}\rho - |\Lambda_n|^{-1/2}x}{\sigma(d)|\Lambda_n|^{1/d}}\right), \tag{13}$$

where $|\Lambda_n| = (2n+1)^d$ is the size of the portfolio $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$. Heuristically, interpolation between sizes of the portfolios Λ_n allows us to define the approximate loss probabilities larger than $x \in \mathbb{R}_+$ for portfolio size $u \in \mathbb{R}_+$ by

$$\Psi_{d,\rho}(u, x) = \Phi\left(\frac{u^{1/2}\rho - u^{-1/2}x}{\sigma(d)u^{1/d}}\right). \tag{14}$$

Hence, losses of a portfolio of u firms are approximately normal with mean ρu and variance $\sigma^2(d)u^{1+2/d}$, that is, the losses of u firms are approximately $\mathcal{N}(\rho u, \sigma^2(d)u^{1+2/d})$. The risk of large losses is captured by the variance of the approximating normal variable. The variance is of order $u^{1+2/d}$. The exponent decreases to 1 as $d \to \infty$.

The interaction of the firms leads to strong dependence of the liquidity states of different firms. We shall compare the results for the distribution ν_ρ to the benchmark case of independent firms. That is, we will assume that the benchmark distribution π_ρ of liquidity states is given by a product of Bernoulli measures with parameter ρ. If we exchange ν_ρ against π_ρ, we have to replace the normalization factor $|\Lambda_n|^{-(d+2)/2d}$ in (11) simply by the usual $|\Lambda_n|^{-1/2}$ and use instead of the limiting variance $\sigma^2(d)$ the quantity $\rho(1-\rho)$. The uniform approximation (12) becomes in this case

$$\sup_{x\in\mathbb{R}_+}\left|\pi_\rho(L_n \geqslant x) - \Phi\left(\frac{|\Lambda_n|^{1/2}\rho - |\Lambda_n|^{-1/2}x}{\sqrt{\rho(1-\rho)}}\right)\right| \leqslant \varepsilon_n, \tag{15}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. For independent firms the speed of convergence to the normal distribution can be bounded by the Berry–Esseen theorem (see e.g. Theorem 4.9 and Remarks in Chapter 2 of Durrett (1996)):

$$\varepsilon_n \leqslant \frac{1 + 2\rho(\rho - 1)}{\sqrt{\rho(1-\rho)}} \frac{1}{(2n)^{d/2}}.$$

By inequality (15) the probability of a loss larger than $x \in \mathbb{R}_+$ can uniformly be approximated by the function

$$\hat{\Psi}_\rho(|\Lambda_n|, x) = \Phi\left(\frac{|\Lambda_n|^{1/2}\rho - |\Lambda_n|^{-1/2}x}{\sqrt{\rho(1-\rho)}}\right). \tag{16}$$

Again interpolation between sizes of the portfolios Λ_n allows us to define the approximate loss probabilities larger than $x \in \mathbb{R}_+$ for portfolio size $u \in \mathbb{R}_+$ by

$$\hat{\Psi}_\rho(u, x) = \Phi\left(\frac{u^{1/2}\rho - u^{-1/2}x}{\sqrt{\rho(1-\rho)}}\right). \tag{17}$$

Hence, losses of a portfolio of u firms are approximately normal with mean ρu and variance $\rho(1-\rho)u$, that is, the losses of u firms are approximately $\mathcal{N}(\rho u, \rho(1-\rho)u)$. In contrast to the contagion case, the order of the variance is simply u.

The order of the variance is related to the riskiness of a portfolio. With contagion, portfolios are more risky than without contagion. In the case of contagion, the order of the variance is $u^{1+2/d}$. The exponent decreases as d increases. Thus, the portfolio becomes more risky if d is small. For $d \leqslant 5$ and reasonable portfolio sizes, say 10,000 firms, this effect cannot be neglected.

To illustrate this, we consider a portfolio of size $u = 10{,}000$ with parameter $\rho = 0.5$, i.e. the marginal probability that a firm is in the low-liquidity state is 0.5. In Figs. 1 and 2 we plot the approximate loss distribution for the benchmark case and the contagion case, where for the latter we vary the dimension d of firm network.

As expected, in comparison with the independence (benchmark) case the loss distribution exhibits a higher variance when credit contagion phenomena are present. Put another way, firm interaction leads to the portfolio being more risky in terms of large losses. With interaction, the probability of exceeding a given loss amount above average losses is larger than in the independence case.

The difference in loss probabilities depends on the dimension. The *higher d*, the *less volatile* is the loss distribution. The approximate loss density for benchmark and interaction case (in dependence of d) is shown in Fig. 3.

While in case $Q = \delta_\rho$ all loss uncertainty averages out in infinite portfolios (cf. (10)), for finite portfolios losses fluctuate around $u\rho = 5000$.

Having investigated the loss distribution in the special case where $Q = \delta_\rho$ for $\rho \in (0, 1)$, we now consider the case of general Q. In this situation the invariant distributions μ of liquidity states are mixtures of the extremal measures ν_ρ, which we focused on in the special case (for a given ρ). Let $\mu = \int_0^1 \nu_\rho Q(\mathrm{d}\rho)$ be an equilibrium liquidity distribution. If Q puts positive mass on 0 or 1, all firms are in the same liquidity state with positive probability. As before, we exclude these trivial cases by

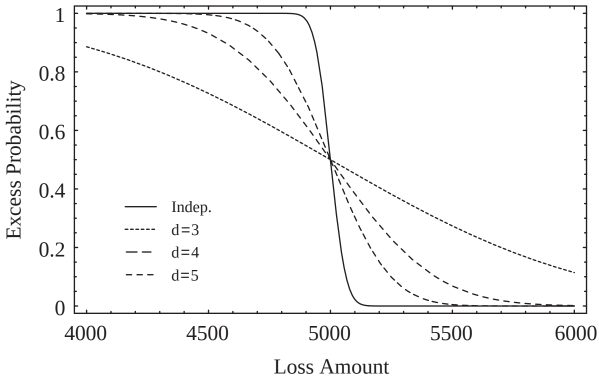

Fig. 1. Probability of a portfolio loss exceeding a given amount, varying d ($u = 10,000$ and $\rho = 0.5$).

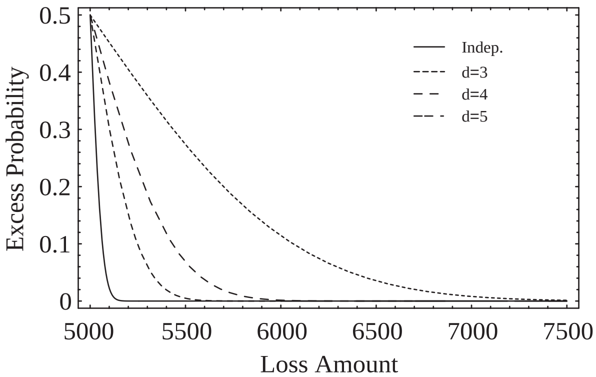

Fig. 2. Probability of a portfolio loss exceeding a given amount, varying d ($u = 10,000$ and $\rho = 0.5$).

assuming $Q(\{0\}) = Q(\{1\}) = 0$. In this general case, the exact probability of a loss larger than $x \in \mathbb{R}_+$ equals

$$\int v_\rho(L_n \geqslant x) Q(\mathrm{d}\rho).$$

In a large portfolio, the law of the losses L_n can be uniformly approximated by a mixture of normal distributions:

Corollary 4.2. *Let $d > 2$ and $M_r = \delta_r$ for $r \in \{0, 1\}$. The distribution of portfolio losses L_n can uniformly be approximated*:

$$\sup_{x \in \mathbb{R}} \left| \int v_\rho(L_n \geqslant x) Q(\mathrm{d}\rho) - \int \Phi\left(\frac{|\Lambda_n|^{1/2}\rho - |\Lambda_n|^{-1/2} x}{\sigma(\rho)|\Lambda_n|^{1/d}} \right) Q(\mathrm{d}\rho) \right| \leqslant \varepsilon_n, \tag{18}$$

where the error bound $\varepsilon_n \to 0$ as $n \to \infty$.

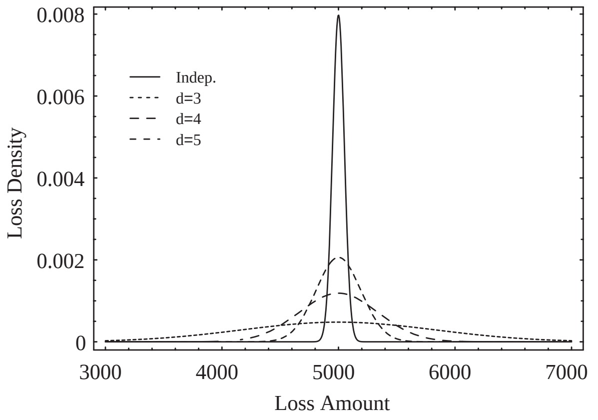

Fig. 3. Approximate density of portfolio losses, varying d ($u = 10,000$ and $\rho = 0.5$).

Proof. See Appendix B. □

Based on this result, in close analogy to (14) interpolation between sizes of the portfolios Λ_n allows us to define the approximate loss probabilities larger than $x \in \mathbb{R}_+$ for portfolio size $u \in \mathbb{R}_+$ by $\int \Phi((u^{1/2}\rho - u^{-1/2}x)/\sigma(\rho)u^{1/d})Q(\mathrm{d}\rho)$. Paralleling (17), in the benchmark case with independent firms the approximate loss probabilities can be defined by $\int \Phi((u^{1/2}\rho - u^{-1/2}x)/\sqrt{\rho(1-\rho)})Q(\mathrm{d}\rho)$, $x, u \in \mathbb{R}_+$.

In both cases—with and without contagion—the systematic risk described by the distribution Q governs the approximate loss distribution. The Gaussian integrands cause additional fluctuations around their random means. If contagion is present, the variance of these Gaussian distributions is of larger order in the number of positions u. The order decreases with increasing dimension of the lattice.

In Fig. 4 we illustrate the approximate density of portfolio losses in the case $Q = 0.4\delta_{0.3} + 0.6\delta_{0.7}$. The portfolio size is again $u = 10,000$.

In infinite portfolios, average losses are governed by the distribution Q and are thus equal to $\rho = 0.3$ with probability 0.4 and equal to $\rho = 0.7$ with probability 0.6. The risk associated with uncertainty about the parameter ρ cannot be reduced by means of ordinary diversification.

In finite portfolios of 10,000 firms, losses fluctuate around $0.3u = 3000$ (with probability 0.4) and $0.7u = 7000$ (with probability 0.6). In analogy to the no-uncertainty case $Q = \delta_\rho$ considered in Fig. 3, interaction leads to more fluctuations when compared to the benchmark case. The degree of additional fluctuation depends on the dimension of the lattice. Nevertheless, the uncertainty about ρ dominates the risk arising from contagion.

In the spirit of the Bernoulli mixture models for dependent defaults (Frey and McNeil, 2003), ρ can be interpreted as the "driving factor" of the model. As we discussed earlier, the macroscopic parameter ρ is not uniquely determined by the

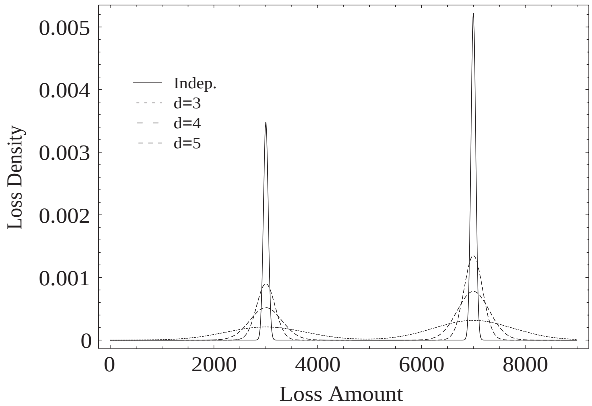

Fig. 4. Approximate density of portfolio losses, varying d ($u = 10,000$ and $Q = 0.4\delta_{0.3} + 0.6\delta_{0.7}$).

local interactions of the firms, since the voter model allows for many invariant measures. Because of this *phase transition*, a risk manager needs to estimate the distribution Q from other data. While the systematic risk associated with ρ cannot be reduced by means of diversification, ρ should be highly correlated with other macroeconomic quantities like GDP growth rates or the level of interest rates. It should therefore be possible to approximately hedge and price the risk associated with ρ by arbitrage arguments.

The residual risk of the portfolio can theoretically be reduced by means of diversification. In practice, however, this is difficult: existing credit exposures cannot simply be reduced or extended nor can new credit exposures simply be added. In this situation *credit derivatives* such as collateralized debt obligations provide an efficient means to achieve the desired diversification on the aggregate level. The (approximate) loss distribution we derive could then be used for the design, pricing and risk management of the credit derivative instruments.

4.2. Stochastic conditional losses

We study the distribution of aggregate portfolio losses L_n in the general case. In a first step we consider the average losses in the portfolio Λ_n. Let $\mu = \int_0^1 v_\rho Q(\mathrm{d}\rho)$ again be an equilibrium distribution of liquidity states where the average number of low-liquidity firms in the whole economy is distributed according to Q. The joint distribution of losses is given by the mixture

$$\beta(\mathrm{d}w) = \int \left(\bigotimes_{i \in \mathbb{Z}^d} M_{\xi(i)} \right)(\mathrm{d}w)\mu(\mathrm{d}\xi), \quad w \in \mathbb{R}^{\mathbb{Z}^d}.$$

By a conditional law of large numbers we have that

$$\lim_{n\to\infty} \frac{L_n}{|\Lambda_n|} =: m(\rho) =: m \tag{19}$$

exists β-almost surely. Writing $m = \rho(l_1 - l_0) + l_0$, we obtain that ρ is random with distribution Q. Due to the ergodicity of the measures v_ρ, in infinite portfolios average losses do not depend on the interaction of firms, but only on systematic risk. Our next result shows that in large portfolios the quantiles $q_\alpha(L_n)$ of the loss distribution are essentially governed by the quantiles of Q.

Proposition 4.3. *Let $q_\alpha(Q)$ be the α-quantile of the distribution Q and assume that the cumulative distribution function G of Q is strictly increasing at $q_\alpha(Q)$, i.e. $G(q_\alpha(Q) + \varepsilon) > \alpha$ and $G(q_\alpha(Q) - \varepsilon) < \alpha$ for every $\varepsilon > 0$. Then*

$$\lim_{n\to\infty} \frac{q_\alpha(L_n)}{|\Lambda_n|} = q_\alpha(Q)(l_1 - l_0) + l_0,$$

where l_r is the expected loss on a position with a firm in liquidity state $r \in \{0, 1\}$. Here, $q_\alpha(L_n)$ denotes an α-quantile of the distribution of L_n under the measure β.

Proof. See Appendix B. □

Frey and McNeil (2003) proved a similar result for exchangeable Bernoulli mixture models, in which credit losses are conditionally independent given some exogenous macro-economic factors. In this context the quantiles of the given factor distribution (the mixing distribution) essentially determine the quantiles of the loss distribution for large homogeneous portfolios. This tail behavior is of central significance for risk measurement and management, as it corresponds to a probabilistic assessment of the scenarios with extremely large losses. Analogously, in our credit contagion approach the tail properties of the systematic risk Q essentially govern the tail behavior of aggregate losses in large portfolios, i.e. the extent of excessive fluctuations of the losses L_∞ in an infinitely large portfolio.

We again focus first on the case $Q = \delta_\rho$ for $\rho \in (0, 1)$, and investigate the distribution of the losses

$$\beta(\mathrm{d}w) = \int \left(\bigotimes_{i\in\mathbb{Z}^d} M_{\xi(i)} \right)(\mathrm{d}w) v_\rho(\mathrm{d}\xi), \quad w \in \mathbb{R}^{\mathbb{Z}^d}.$$

As in the case of deterministic conditional losses, for large portfolios the law of the losses L_n can again be approximated by a normal distribution. In this case, the expected loss equals m as defined in (19) and can be written as

$$m = \rho(l_1 - l_0) + l_0.$$

Theorem 4.4. *Let $d > 2$ and suppose that $Q = \delta_\rho$ for $\rho \in (0, 1)$. For large portfolios the distribution of losses satisfies*

$$|\Lambda_n|^{-(d+2)/2d}(L_n - |\Lambda_n| m) = |\Lambda_n|^{-(d+2)/2d} \sum_{i\in\Lambda_n} (U(i) - m) \xrightarrow{\mathrm{w}} \mathcal{N}(0, (l_1 - l_0)^2\sigma^2),$$

where σ^2 *denotes the limiting variance* (27), *cf. Appendix C. The loss distribution can uniformly be approximated*:

$$\sup_{x\in\mathbb{R}}\left|\beta(L_n\geqslant x)-\Phi\left(\frac{|\Lambda_n|^{1/2}m-|\Lambda_n|^{-1/2}x}{(l_1-l_0)\sigma|\Lambda_n|^{1/d}}\right)\right|\leqslant\varepsilon_n, \tag{20}$$

where $\varepsilon_n\to 0$ *as* $n\to\infty$.

Proof. See Appendix B. □

Based on inequality (20), interpolation between sizes of the portfolios Λ_n allows us to define the approximate loss probabilities larger than $x\in\mathbb{R}_+$ for portfolio size $u\in\mathbb{R}_+$ by $\Phi((u^{1/2}m-u^{-1/2}x)/(l_1-l_0)\sigma u^{1/d})$. This result corresponds to formula (14), which we obtained in the case with deterministic conditional losses. In case of stochastic conditional losses the limiting variance is multiplied by the factor $(l_1-l_0)^2$, which depends only on the expected value of the loss distributions M_r, $r\in\{0,1\}$. Because of the non-classical re-scaling, the random fluctuations of the distributions M_r are averaged out in the normal approximation.

In analogy to Corollary 4.2, we extend our analysis of the loss distribution to general invariant distributions μ of liquidity states, which are mixtures of the extremal measures we have considered so far. The joint distribution of the losses is given by the mixture

$$\beta(\mathrm{d}w)=\int\left(\bigotimes_{i\in\mathbb{Z}^d}M_{\xi(i)}\right)(\mathrm{d}w)\mu(\mathrm{d}\xi),\quad w\in\mathbb{R}^{\mathbb{Z}^d}.$$

Corollary 4.5. *Let* $d>2$. *For a large portfolio, the distribution of losses* L_n *can uniformly be approximated*:

$$\sup_{x\in\mathbb{R}}\left|\beta(L_n\geqslant x)-\int\Phi\left(\frac{|\Lambda_n|^{1/2}m-|\Lambda_n|^{-1/2}x}{(l_1-l_0)\sigma(\rho)|\Lambda_n|^{1/d}}\right)Q(\mathrm{d}\rho)\right|\leqslant\varepsilon_n, \tag{21}$$

where $m=\rho(l_1-l_0)+l_0$ *and* $\varepsilon_n\to 0$ *as* $n\to\infty$.

Proof. See Appendix B. □

Compared with inequality (18), if conditional losses are stochastic the approximate variance $\sigma^2(\rho)$ is multiplied by a factor $(l_1-l_0)^2$ and the averages of low-liquidity states ρ are replaced by m. Qualitatively, the approximate loss distribution has similar properties in both cases (18) and (21). Interestingly, the fluctuations of the distributions M_r around their means are averaged out in the normal approximation; only the expectations enter inequality (21).

5. Discussion

Credit contagion refers to the propagation of economic distress from one firm to another. A thorough understanding of contagion processes is essential for the management and regulation of financial institutions. In this paper we model credit contagion phenomena and study their effects on the volatility of losses on large portfolios of financial positions. We derive an explicit analytical approximation to the distribution of losses on large portfolios of financial positions whose issuers are subject to credit contagion.

Our contagion model is stylized. The economy is modeled by a multi-dimensional lattice, whose nodes are identified with firms. The edges represent business partner relationships. Firms are homogeneous in their individual characteristics. While they may be in different states, they have the same number of business partner relationships and are of equal size. Furthermore, they carry the same amount of marginal risk. It seems difficult to relax this homogeneity assumption when explicit approximation results are desired.

The business partner relationships are the channel for credit contagion phenomena, i.e. the propagation of liquidity shocks through a network of obligations. The direction in which shocks are propagated is symmetric. The likelihood of a healthy firm to become distressed increases with the number of distressed business partners. Vice versa, the likelihood of a distressed firm to make a turnaround increases with the number of healthy business partners. Less realistic is the symmetry in the business relationships: any two firms influence each other to the same degree. Here asymmetry could be accounted for by considering a directed graph. Explicit analytical results are hard to come by in the asymmetric case. If numerical results suffice, then the direct simulation of the contagion processes and the associated losses is straightforward; see Egloff et al. (2004) for such a simulation study.

The contagion dynamics are described by the basic voter process, which is well studied in the theory of interacting particle systems. Our explicit approximation results are based on the recent results of Zähle (2001), who proves a non-standard central limit theorem for the voter model in equilibrium. The contact process is an alternative choice for the contagion dynamics. It allows for asymmetric interaction, but it is difficult to obtain an analogous approximation result.

Two further issues related to our Gaussian approximation are left open for future research. The first concerns the speed of convergence of the voter process to its equilibrium. The second is the characterization of the approximation error in dependence of the portfolio size.

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Appendix A. Ergodic decomposition

This appendix provides supplementary results, and proofs of Theorem 3.3 and Corollary 3.4. Let $X = \{0,1\}^{\mathbb{Z}^d}$. By $\mathscr{M}_e$ we denote the space of ergodic probability measures on X endowed with weak topology. Let $\mathscr{G}$ be the Borel σ-algebra on $\mathscr{M}_e$. We write $\mathscr{M}_{e,\rho}$ for the subspace of $\mathscr{M}_e$ of probability measures v with $v\{\xi: \xi(0) = 1\} = \rho \in [0,1]$. The theorem of Choquet states that any translation-invariant probability measure on X can be represented as a mixture of ergodic measures (Georgii, 1988, Theorem 14.10):

Theorem A.1. *Let μ be a translation-invariant probability measure on X. Then there exists a probability measure $\hat{\gamma}$ on $\mathscr{M}_e$ such that $\mu = \int_{\mathscr{M}_e} v\hat{\gamma}(\mathrm{d}v)$, i.e. for all continuous functions $f \in C(X)$ it holds that $\mu(f) = \int_{\mathscr{M}_e} v(f)\hat{\gamma}(\mathrm{d}v)$.*

The following theorem refines the statement of Choquet and follows from Theorem A.1 and Doob's functional representation. A detailed proof can be found in Giesecke and Weber (2004a).

Theorem A.2. *Let μ be a translation-invariant probability measure on X. Then there exists a probability measure Q on $[0,1]$ and a kernel*

$$\gamma_{\cdot}(\cdot): \begin{cases} \mathscr{G} \times [0,1] & \to & [0,1], \\ (A, \rho) & \mapsto & \gamma_\rho(A) \end{cases}$$

with $\gamma_\rho(\mathscr{M}_{e,\rho}) = 1$ such that

$$\mu = \int_{[0,1]} \left(\int_{\mathscr{M}_e} v\gamma_\rho(\mathrm{d}v) \right) Q(\mathrm{d}\rho).$$

Let $\hat{\gamma}$ be defined as in Theorem A.1. Q has the cumulative distribution function G given by

$$G(\rho) = \hat{\gamma}\{v \in \mathscr{M}_e: v\{\xi: \xi(0) = 1\} \leqslant \rho\}.$$

We are now in a position to prove Theorem 3.3 in the text:

Proof of Theorem 3.3. Let $f \in C(X)$. Writing μ_t^ξ instead of $\mu_t^{\delta_\xi}$, we have

$$\mu_t^\kappa(f) = \int \mu_t^\xi(f)\kappa(\mathrm{d}\xi) = \int_0^1 \left(\int_{\mathscr{M}_e} \left(\int \mu_t^\xi(f) v(\mathrm{d}\xi) \right) \gamma_\rho(\mathrm{d}v) \right) Q(\mathrm{d}\rho)$$
$$= \int_0^1 \left(\int_{\mathscr{M}_e} \mu_t^v(f)\gamma_\rho(\mathrm{d}v) \right) Q(\mathrm{d}\rho).$$

Moreover, by the bounded convergence theorem

$$\lim_{t\to\infty} \mu_t^\kappa(f) = \lim_{t\to\infty} \int_0^1 \left(\int_{\mathscr{M}_e} \mu_t^v(f)\gamma_\rho(\mathrm{d}v) \right) Q(\mathrm{d}\rho)$$
$$= \int_0^1 \left(\int_{\mathscr{M}_e} \left(\lim_{t\to\infty} \mu_t^v(f) \right) \gamma_\rho(\mathrm{d}v) \right) Q(\mathrm{d}\rho)$$

since $|\mu_t^\nu(f)| \leqslant \|f\|_\infty < \infty$. Noting that $\lim_{t\to\infty}\mu_t^\nu(f) = \nu_\rho(f)$ on $\mathcal{M}_{e,\rho}$ and that $\gamma_\rho(\mathcal{M}_{e,\rho}) = 1$, we have $\lim_{t\to\infty}\mu_t^\kappa(f) = \int_0^1 \nu_\rho(f)Q(\mathrm{d}\rho)$. □

Proof of Corollary 3.4. Part (1) can be verified as follows. For $t = \infty$ the claim holds by a strong law of large numbers, since ν_ρ is ergodic with $\nu_\rho\{\xi\colon \xi(i) = 1\} = \rho$. Let $\kappa_{\rho,t}$ be given for $t \in [0,\infty)$. According to Theorem A.2 there exists a kernel γ' and a measure Q' on $[0,1]$ such that

$$\kappa_{\rho,t} = \int_{[0,1]}\left(\int_{\mathcal{M}_e} \nu\gamma'_{\rho'}(\mathrm{d}\nu)\right)Q'(\mathrm{d}\rho').$$

Suppose that $\kappa_{\rho,t}(\bar\rho = \rho) < 1$. Suppose also that $Q' = \delta_\rho$. Thus,

$$\kappa_{\rho,t}(\bar\rho = \rho) = \left(\int_{\mathcal{M}_e} \nu\gamma'_\rho(\mathrm{d}\nu)\right)(\bar\rho = \rho) = \int_{\mathcal{M}_{e,\rho}} \nu(\bar\rho = \rho)\gamma'_\rho(\mathrm{d}\nu) = 1,$$

a contradiction. Thus, $Q' \neq \delta_\rho$. Hence by Theorem 3.3,

$$\nu_\rho = \lim_{t\to\infty}\kappa_{\rho,t} = \lim_{t\to\infty}\int_{[0,1]}\left(\int_{\mathcal{M}_e} \nu\gamma'_{\rho'}(\mathrm{d}\nu)\right)Q'(\mathrm{d}\rho') = \int_{[0,1]} \nu_{\rho'}Q'(\mathrm{d}\rho'),$$

a contradiction. Thus, $\kappa_{\rho,t}(\bar\rho = \rho) = 1$.

Part (2) can be proven as follows:

$$\begin{aligned}\mu_t^\kappa(\bar\rho \leqslant \rho') &= \int_{[0,1]} \kappa_{\rho,t}(\bar\rho \leqslant \rho')Q(\mathrm{d}\rho) = \int_{[0,1]} \mathbf{1}_{(-\infty,\rho']}(\rho)Q(\mathrm{d}\rho)\\ &= Q(-\infty,\rho'). \qquad \square\end{aligned}$$

Appendix B. Normal approximation

This appendix is devoted to the proof of Theorems 4.1 and 4.4, Corollaries 4.2 and 4.5 as well as Proposition 4.3. An integral formula for the limiting variance $\sigma^2 = \sigma^2(d)$ is given in Appendix C and involves the escape probability γ_d of a random walk. We start by considering γ_d.

Theorem B.1. *Let Y_n be a simple random walk on $\mathbb{Z}^d$ with $d \geqslant 3$. The escape probability γ_d can be calculated by*

$$\gamma_d = \frac{1}{J(d)}, \tag{22}$$

where the quantity $J(d)$ is defined by

$$J(d) = (2\pi)^{-d} \int_{(-\pi,\pi)^d} \left(1 - \frac{1}{d} \sum_{m=1}^{d} \cos x_m\right)^{-1} \mathrm{d}x. \tag{23}$$

Numerical values are given in the following table:

d	3	4	5	6
$J(d)$	1.516386	1.239467	1.156308	1.116963
γ_d	0.659463	0.806798	0.864821	0.895285

Proof of Theorem B.1. A detailed proof of the theorem which uses standard arguments from the theory of random walks (see e.g. Chapter 3 of Durrett (1996)) can be found in Giesecke and Weber (2004a). In order to obtain the numerical values in the table observe that $J(d) = d \cdot I(d;1) = L(d;1) + 1$ where the functions I and L are defined and evaluated in Kondo and Hara (1987). □

Proof of Theorem 4.1. The normal approximation result can be derived from Theorem 1 in Zähle (2001). For a detailed proof see Giesecke and Weber (2004a).

Next we derive the uniform approximation (12). Since the distribution function of the normal distribution is continuous, it follows from Exercise 2.6. in Chapter 2 of Durrett (1996) that

$$\sup_{z\in\mathbb{R}} \left| \nu_\rho \left(\frac{|\Lambda_n|^{-(d+2)/2d}(L_n - |\Lambda_n|\rho)}{\sigma} \geqslant z \right) - \Phi(-z) \right| \leqslant \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. The approximation (12) follows immediately, since

$$\{L_n \geqslant x\} = \left\{ \frac{|\Lambda_n|^{-(d+2)/2d}(L_n - |\Lambda_n|\rho)}{\sigma} \geqslant \frac{|\Lambda_n|^{-(d+2)/2d}(x - |\Lambda_n|\rho)}{\sigma} \right\}.$$

σ is given in Appendix C. □

Proof of Corollary 4.2. The distribution of $|\Lambda_n|^{-(d+2)/2d} \sum_{i\in\Lambda_n} (\xi(i) - \rho)$ under the measure ν_ρ will be denoted by ς_ρ^n. We define the quantity

$$\delta_\rho^n := \sup_{n' \geqslant n} \sup_{z\in\mathbb{R}} \left| \varsigma_\rho^{n'}([z,\infty)) - \Phi\left(-\frac{z}{\sigma(\rho)}\right) \right|.$$

Arguing as in the previous proof, we see that Theorem 1 in Zähle (2001) and Exercise 2.6 in Chapter 2 of Durrett (1996) imply that δ_ρ^n converges to 0 for all $\rho \in (0,1)$ as $n \to \infty$. Observe that $\rho \mapsto \delta_\rho^n$ is measurable. For $\varepsilon > 0$ we can therefore define measurable sets

$$A_\varepsilon^n = \{\rho \in (0,1) : \delta_\rho^n < \varepsilon\}.$$

Then $A_\varepsilon^n \subseteq A_\varepsilon^{n+1}$, and $Q(A_\varepsilon^n) \nearrow 1$ as $n \to \infty$. Choose n_0 large enough such that

$$Q(A_\varepsilon^{n_0}) \geqslant 1 - \varepsilon.$$

Let $\rho \mapsto z(\rho)$ be a measurable mapping. Then for all $n \geqslant n_0$ we get

$$\begin{aligned} &\left| \int \left[\varsigma_\rho^n([z(\rho), \infty)) - \Phi\left(-\frac{z(\rho)}{\sigma(\rho)} \right) \right] Q(\mathrm{d}\rho) \right| \\ &\quad \leqslant 2(1 - Q(A_\varepsilon^n)) + \sup_{\rho \in A_\varepsilon^n} \sup_{z' \in \mathbb{R}} \left| \varsigma_\rho^n([z', \infty)) - \Phi\left(-\frac{z'}{\sigma(\rho)} \right) \right| \leqslant 3\varepsilon. \end{aligned}$$

Let $x \in \mathbb{R}$ be arbitrary, and let $n \geqslant n_0$. We can choose

$$z(\rho) = |\Lambda_n|^{-(d+2)/2d}(x - |\Lambda_n|\rho).$$

It follows that for any $x \in \mathbb{R}$ and $n \geqslant n_0$ the following inequality holds:

$$\left| \int v_\rho(L_n \geqslant x) Q(\mathrm{d}\rho) - \int \Phi\left(\frac{|\Lambda_n|^{1/2}\rho - |\Lambda_n|^{-1/2}x}{\sigma(\rho)|\Lambda_n|^{1/d}} \right) Q(\mathrm{d}\rho) \right| \leqslant 3\varepsilon. \qquad \square$$

Proof of Proposition 4.3. For $\rho \in [0, 1]$ define the probability measures

$$\beta_\rho(\mathrm{d}w) = \int \left(\bigotimes_{i \in \mathbb{Z}^d} M_{\xi(i)} \right)(\mathrm{d}w) v_\rho(\mathrm{d}\xi), \quad w \in \mathbb{R}^{\mathbb{Z}^d}.$$

First observe that, due to (19),

$$\lim_{n\to\infty} \beta_\rho \left(\frac{L_n/|\Lambda_n| - l_0}{l_1 - l_0} \leqslant a \right) = \begin{cases} 1, & \rho < a, \\ 0, & \rho > a. \end{cases}$$

Let $\varepsilon > 0$ and let G be the cumulative distribution function of Q. Then

$$\begin{aligned} &\limsup_{n\to\infty} \beta\{L_n - l_0|\Lambda_n| \leqslant |\Lambda_n|(l_1 - l_0)(q_\alpha(Q) - \varepsilon)\} \\ &\quad = \limsup_{n\to\infty} \int_0^1 \beta_\rho\{L_n - l_0|\Lambda_n| \leqslant |\Lambda_n|(l_1 - l_0)(q_\alpha(Q) - \varepsilon)\}\,\mathrm{d}G(\rho) \\ &\quad \leqslant \int_0^1 \limsup_{n\to\infty} \beta_\rho \left(\frac{L_n/|\Lambda_n| - l_0}{l_1 - l_0} \leqslant q_\alpha(Q) - \varepsilon \right) \mathrm{d}G(\rho) \\ &\quad \leqslant \int_0^1 1_{(-\infty, q_\alpha(Q) - \varepsilon]}(\rho)\,\mathrm{d}G(\rho) = G(q_\alpha(Q) - \varepsilon) < \alpha, \end{aligned}$$

where the last equality is strict by assumption. The first inequality follows from Fatou's lemma. Analogously,

$$\liminf_{n\to\infty} \beta\{L_n - l_0|\Lambda_n| \leqslant |\Lambda_n|(l_1 - l_0)(q_\alpha(Q) + \varepsilon)\} \geqslant G(q_\alpha(Q) + \varepsilon/2) > \alpha.$$

Hence, for n large enough,

$$|\Lambda_n|(l_1 - l_0)(q_\alpha(Q) - \varepsilon) \leqslant q_\alpha(L_n - l_0|\Lambda_n|) \leqslant |\Lambda_n|(l_1 - l_0)(q_\alpha(Q) + \varepsilon).$$

The claim follows from observing that $q_\alpha(L_n - l_0|\Lambda_n|) = q_\alpha(L_n) - l_0|\Lambda_n|$. $\square$

Proof of Theorem 4.4. This is a corollary of the normal approximation results in the deterministic case. Define the function $f: \{0,1\} \to \{l_0, l_1\}$ by $f(0) = l_0$ and $f(1) = l_1$. f is used to introduce the random variables $m_i = f(\xi(i))$, $i \in \mathbb{Z}^d$. It is easy to see that (11) implies

$$|\Lambda_n|^{-(d+2)/2d} \sum_{i \in \Lambda_n} (m_i - m) \xrightarrow{\text{w}} \mathcal{N}(0, (l_1 - l_0)^2 \sigma^2). \tag{24}$$

Denote now by $(X_{r,i})_{i \in \mathbb{Z}^d}$ independent random variables with distribution M_r, $r \in \{0,1\}$. Then we can rewrite the renormalized losses as

$$\begin{aligned} |\Lambda_n|^{-(d+2)/2d}(L_n - |\Lambda_n| m) = {} & |\Lambda_n|^{(d+2)/2d} \sum_{i \in \Lambda_n, \xi(i)=0} (X_{0,i} - m_i) \\ & + |\Lambda_n|^{-(d+2)/2d} \sum_{i \in \Lambda_n, \xi(i)=1} (X_{1,i} - m_i) \\ & + |\Lambda_n|^{-(d+2)/2d} \sum_{i \in \Lambda_n} (m_i - m). \end{aligned}$$

The last summand on the right-hand side converges weakly according to (24). The other two terms converge almost surely to 0; w.l.o.g. we will prove this fact only for the first term, i.e.

$$|\Lambda_n|^{-(d+2)/2d} \sum_{i \in \Lambda_n, \xi(i)=0} (X_{0,i} - m_i) = |\Lambda_n|^{-(d+2)/2d} \sum_{i \in \Lambda_n, \xi(i)=0} (X_{0,i} - l_0). \tag{25}$$

The random number of summands in (25) equals $c(n) = |\{i \in \Lambda_n: \xi(i) = 0\}|$ and is almost surely increasing to ∞ as $n \to \infty$. Theorem 8.7. of Chapter 1 in Durrett (1996) implies for $\varepsilon > 0$ that

$$c(n)^{-1/2} (\log c(n))^{-(1/2+\varepsilon)} \sum_{i \in \Lambda_n, \xi(i)=0} (X_{0,i} - l_0) \tag{26}$$

converges to 0 as $n \to \infty$. The last result can also be viewed as a consequence of the law of iterated logarithm.

Now observe that for $\varepsilon > 0$ the sequence $c(n)$ satisfies

$$\frac{c(n)^{1/2} (\log c(n))^{1/2+\varepsilon}}{|\Lambda_n|^{(d+2)/2d}} \leqslant \frac{|\Lambda_n|^{1/2} (\log |\Lambda_n|)^{1/2+\varepsilon}}{|\Lambda_n|^{(d+2)/2d}} = \frac{(\log |\Lambda_n|)^{1/2+\varepsilon}}{|\Lambda_n|^{1/d}}.$$

The last term converges to 0 as $n \to \infty$. This fact together with (26) implies that the terms in (25) converge to 0 as $n \to \infty$.

Altogether we obtain for $n \to \infty$ the weak convergence,

$$|\Lambda_n|^{-(d+2)/2d}(L_n - |\Lambda_n|m) \xrightarrow{\text{w}} \mathcal{N}(0, (l_1 - l_0)^2\sigma^2).$$

The uniform approximation (20) is obtained with the same arguments as in the deterministic case. □

Proof of Corollary 4.5. Analogous to the proof of Corollary 4.2. □

Appendix C. Limiting variance

The limiting variance $\sigma^2 = \sigma^2(d)$ is given by

$$\sigma^2 = \rho(1-\rho)\frac{\gamma_d d}{2^{d+3}\pi^{d/2}}\,\Gamma\left(\frac{d-2}{2}\right)\int_{[-1,1]^d}\int_{[-1,1]^d}\frac{1}{\|x-y\|_2^{d-2}}\,\mathrm{d}x\,\mathrm{d}y. \tag{27}$$

Here, Γ is the Gamma-function and $\gamma = \gamma_d$ is given by Theorem B.1.

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