Time consistency of multivariate dynamic risk measures

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Outline

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1. Set-valued risk measures: Setup

- Probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- d assets (may include different currencies)
- Portfolio vectors in physical units (numéraire free), i.e. number of units in d assets
- Claim: $X \in L^p_d(\mathcal{F}_T)$ payoff (in physical units) at time T
- Convex transaction costs at time t: closed convex set $\mathbb{R}^d_+ \subseteq K_t[\omega] \subseteq \mathbb{R}^d$ (solvency region), positions transferable into non-negative portfolios
- Eligible portfolios M, linear subspace of $\mathbb{R}^d$ of portfolios that can be used to compensate risk (e.g. Dollars & Euros)

1.1 Set-valued risk measures: Primal representation

- $M_t := L^p_d(\mathcal{F}_t; M), \quad M_{t,+} := M_t \cap L^p_d(\mathcal{F}_t)_+$
- $\mathcal{P}(\mathcal{Z}; C) := \{A \subseteq \mathcal{Z} : A = A + C\}$
- $\mathcal{G}(\mathcal{Z}; C) := \{A \subseteq \mathcal{Z} : A = \operatorname{cl}\operatorname{co}(A + C)\}$

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Conditional Set-Valued Risk Measure

A set-valued function $R_t : L^p_d(\mathcal{F}_T) \to \mathcal{P}(M_t; M_{t,+})$ is a conditional risk measure if

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1. Finite at zero: $\emptyset \neq R_t(0) \neq M_t$;
2. M_t translative: $R_t(X + m) = R_t(X) - m$ for any $m \in M_t$;
3. $L_d^p(\mathcal{F}_T)_+$ monotone: if $X - Y \in L_d^p(\mathcal{F}_T)_+$ then $R_t(X) \supseteq R_t(Y)$.

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- ***Normalized***: for every $X \in L^p_d(\mathcal{F}_t) : R_t(X) = R_t(X) + R_t(0)$.
- Normalized version:
  $\bar{R}_t(X) := R_t(X) -^{\cdot} R_t(0) = \{u \in M_t : R_t(0) + u \subseteq R_t(X)\}$.

- ***(Conditionally) convex***: for all $X, Y \in L^p_d(\mathcal{F}_T)$, for all $0 \leq \lambda \leq 1$ ($\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$)

$$R_t(\lambda X + (1-\lambda)Y) \supseteq \lambda R_t(X) + (1-\lambda)R_t(Y).$$

- ***(Conditionally) positive homogeneous***: for all $X \in L^p_d(\mathcal{F}_T)$, for all $\lambda > 0$ ($\lambda \in L^\infty(\mathcal{F}_t)_{++}$)

$$R_t(\lambda X) = \lambda R_t(X).$$

- ***(Conditionally) coherent***: if it is (conditionally) convex and (conditionally) positive homogeneous.
- K***-compatible*** for some set $K \subseteq L^p_d(\mathcal{F}_T)$ if there exists a risk measure $\tilde{R}$ such that $R_t(X) = \bigcup_{k \in K} \tilde{R}_t(X - k)$.

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- Acceptance set: $A_t = \left\{X \in L_d^p(\mathcal{F}_T) : 0 \in R_t(X)\right\}$
- Risk measure: $R_t(X) = \{u \in M_t : X + u \in A_t\}$

Properties	
Risk measure	**Acceptance set**
(Conditionally) convex	(Conditionally) convex
(Conditionally) coherent	(Conditionally) convex cone
Closed graph	Closed
$B \subseteq L_d^p(\mathcal{F}_T)$ B-monotone	$A_t + B = A_t$
$C \subseteq M_t$ $R_t(X) : L_d^p(\mathcal{F}_T) \to \mathcal{P}(M_t; C)$	$A_t + C \subseteq A_t$
$R_t(X) \neq \emptyset \; \forall X \in L_d^p(\mathcal{F}_T)$	$L_d^p(\mathcal{F}_T) = A_t + M_t$
$R_t(X) \neq M_t \; \forall X \in L_d^p(\mathcal{F}_T)$	$L_d^p(\mathcal{F}_T) = (L_d^p(\mathcal{F}_T) \backslash A_t) + M_t$

- $f : \mathcal{X} \to \mathcal{G}(\mathcal{Z}; C) = \{D \subseteq \mathcal{Z} : D = \operatorname{cl}\operatorname{co}(D + C)\}$.
- Set-valued infimum: $\inf_{x \in A} f(x) := \operatorname{cl}\operatorname{co} \bigcup_{x \in A} f(x)$;
- Set-valued supremum: $\sup_{x \in A} f(x) := \bigcap_{x \in A} f(x)$.

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- $F^{\mathcal{Z}}_{(x^*,z^*)} : \mathcal{X} \to 2^{\mathcal{Z}}$ for $x^* \in \mathcal{X}^*$ and $z^* \in \mathcal{Z}^*$

$$F^{\mathcal{Z}}_{(x^*,z^*)}(x) := \left\{z \in \mathcal{Z} : \langle x^*, x \rangle \leq \langle z^*, z \rangle \right\}.$$

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1.2 Set-valued risk measures: Dual representation

- $F^{\mathcal{Z}}_{(x^*,z^*)}(x) := \{z \in \mathcal{Z} : \langle x^*, x\rangle \leq \langle z^*, z\rangle\}$.
- (Negative) convex conjugate:

$$
\begin{aligned}
-f^*(x^*, z^*) &= \inf_{x \in \mathcal{X}} \left[F^{\mathcal{Z}}_{(x^*,z^*)}(-x) + f(x)\right] \\
&= \operatorname{cl}\operatorname{co} \bigcup_{x \in \mathcal{X}} \left[F^{\mathcal{Z}}_{(x^*,z^*)}(-x) + f(x)\right].
\end{aligned}
$$

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- (Negative) convex conjugate:

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- Biconjugate:

$$f^{**}(x) = \sup_{(x^*,z^*) \in \mathcal{X}^* \times C^+ \backslash \{0\}} \left[-f^*(x^*, z^*) + F^{\mathcal{Z}}_{(x^*,z^*)}(x)\right] \\ = \bigcap_{(x^*,z^*) \in \mathcal{X}^* \times C^+ \backslash \{0\}} \left[-f^*(x^*, z^*) + F^{\mathcal{Z}}_{(x^*,z^*)}(x)\right].$$

- f is ***proper*** if $f(x) \neq \mathcal{Z}$ for every $x \in \mathcal{X}$ and $f(x) \neq \emptyset$ for some $x \in \mathcal{X}$
- f is ***closed*** if the graph of f ($\operatorname{graph} f = \{(x,z) \in \mathcal{X} \times \mathcal{Z} : \; z \in f(x)\}$) is closed.
- Set-valued Fenchel-Moreau Theorem:

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- f is ***closed*** if the graph of f $(\operatorname{graph} f = \{(x,z) \in \mathcal{X} \times \mathcal{Z} : \; z \in f(x)\})$ is closed.
- Set-valued Fenchel-Moreau Theorem:
  A proper function $f : \mathcal{X} \to \mathcal{G}(\mathcal{Z}; C)$ is closed and convex if and only if $f(x) = f^{**}(x)$ for every $x \in \mathcal{X}$.

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- $\mathcal{Z} = M_t$ and $\mathcal{Z}^* = L^q_d(\mathcal{F}_t)$

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- $\mathcal{Z} = M_t$ and $\mathcal{Z}^* = L^q_d(\mathcal{F}_t)$
- Dual variables:

$$\mathcal{W}_t := \Big\{(\mathbb{Q}, w) \in \mathcal{M}_d \times \left(M_{t,+}^{+} \backslash M_t^{\perp}\right) : \\ \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t},\ w_t^T(\mathbb{Q}, w) \in L^q_d(\mathcal{F}_T)_+\Big\};$$

where $w_t^s(\mathbb{Q}, w) = w \cdot \mathbb{E}_s\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \left(w_1 \mathbb{E}_s\left[\frac{d\mathbb{Q}_1}{d\mathbb{P}}\right], \ldots, w_d \mathbb{E}_s\left[\frac{d\mathbb{Q}_d}{d\mathbb{P}}\right]\right)^{\mathsf{T}}$.

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- $G_t(w) := \left\{u \in L^p_d(\mathcal{F}_t) : \mathbb{E}\left[w^{\mathsf{T}} u\right] \geq 0\right\}$

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- $G_t(w) := \left\{u \in L^p_d(\mathcal{F}_t) : \mathbb{E}\left[w^\mathsf{T} u\right] \geq 0\right\}$
- $F^M_{(\mathbb{Q},w)}(X) = \left\{u \in M_t : \mathbb{E}\left[w^\mathsf{T} u\right] \geq \mathbb{E}\left[w^\mathsf{T}\mathbb{E}^\mathbb{Q}_t\left[X\right]\right]\right\} = \left(\mathbb{E}^\mathbb{Q}_t\left[X\right] + G_t(w)\right) \cap M_t$.

Convex Risk Measures

A function $R_t : L_d^p(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ is a ***closed convex risk measure*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t} \left[-\beta_t^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}_t^{\mathbb{Q}}[-X] + G_t(w)\right) \cap M_t\right],$$

where $-\beta_t^{\min}$ is the minimal penalty function given by

$$-\beta_t^{\min}(\mathbb{Q}, w) = \mathrm{cl} \bigcup_{Z\in A_t} \left(\mathbb{E}_t^{\mathbb{Q}}[Z] + G_t(w)\right) \cap M_t.$$

1.2 Set-valued risk measures: Dual representation

Coherent Risk Meausres

A function $R_t : L_d^p(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ is a ***closed coherent risk measure*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t^{\max}} \left(\mathbb{E}_t^{\mathbb{Q}}\left[-X\right] + G_t\left(w\right)\right) \cap M_t,$$

where $\mathcal{W}_t^{\max}$ is the maximal set of dual variables given by

$$\mathcal{W}_t^{\max} = \left\{(\mathbb{Q}, w) \in \mathcal{W}_t : w_t^T\left(\mathbb{Q}, w\right) \in A_t^+\right\}.$$

1.2 Set-valued risk measures: Dual representation

- $\Gamma_t(w) := \left\{u \in L^p_d(\mathcal{F}_t) : w^\mathsf{T} u \geq 0 \text{ a.s.}\right\}$

Conditionally Convex and Coherent Risk Measures

A function $R_t : L^p_d(\mathcal{F}_T) \to \mathcal{G}(M_t; M_{t,+})$ is a ***closed conditionally convex risk measure*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t} \left[-\alpha_t^{\min}(\mathbb{Q},w) + \left(\mathbb{E}_t^{\mathbb{Q}}\left[-X\right] + \Gamma_t(w)\right) \cap M_t\right],$$

where $-\alpha_t^{\min}$ is the conditional penalty function given by

$$-\alpha_t^{\min}(\mathbb{Q},w) = \operatorname{cl} \bigcup_{Z\in A_t} \left(\mathbb{E}_t^{\mathbb{Q}}\left[Z\right] + \Gamma_t(w)\right) \cap M_t.$$

R_t is additionally ***conditionally coherent*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t^{\max}} \left(\mathbb{E}_t^{\mathbb{Q}}\left[-X\right] + \Gamma_t\left(w\right)\right) \cap M_t.$$

Multi-Portfolio Time Consistency

A dynamic risk measure $(R_t)_{t=0}^T$ is ***multi-portfolio time consistent*** if the relation

$$R_s(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_t(Y)$$

for any times $t < s$, any $X \in L_d^p(\mathcal{F}_T)$ and any $\mathbf{Y} \subseteq L_d^p(\mathcal{F}_T)$

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- Multi-portfolio time consistency implies "time consistency" defined by

  $$R_s(X) \subseteq R_s(Y) \Rightarrow R_t(X) \subseteq R_t(Y)$$

  for any times $t < s$ and $X, Y \in L_d^p(\mathcal{F}_T)$.

Multi-Portfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized ($R_t(X) = R_t(X) + R_t(0)$ for every X and t) then the following are equivalent:

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- $R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X))$;

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- $R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X))$;
- $A_t = A_s + A_{t,s}$ where $A_{t,s} := A_t \cap M_s$.

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- $R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X))$;
- $A_t = A_s + A_{t,s}$ where $A_{t,s} := A_t \cap M_s$.

- If discrete time $t, s \in \{0, 1, ..., T\}$ then sufficient to have any of these conditions with $s = t + 1$.

Stepped Risk Measures

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A ***stepped risk measure*** $R_{t,s} : M_s \to \mathcal{P}(M_t; M_{t,+})$ from time t to s is the restriction of a risk measure R_t to M_s.

- If $R_{t,s}$ is closed and convex (coherent, conditionally convex) then it can be defined by the penalty function (dual set, conditional penalty function)

$$-\beta_{t,s}^{\min}(\mathbb{Q}, w) := \text{cl} \bigcup_{Z \in A_{t,s}} \left(\mathbb{E}_t^{\mathbb{Q}}[Z] + G_t(w)\right) \cap M_t$$

$$\mathcal{W}_{t,s}^{\max} := \{(\mathbb{Q}, w) \in \mathcal{W}_{t,s} : w_t^s(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_s)_+\}$$

$$-\alpha_{t,s}^{\min}(\mathbb{Q}, w) := \text{cl} \bigcup_{Z \in A_{t,s}} \left(\mathbb{E}_t^{\mathbb{Q}}[Z] + \Gamma_t(w)\right) \cap M_t$$

over the set of stepped dual variables

$$\mathcal{W}_{t,s} := \Big\{(\mathbb{Q}, w) \in \mathcal{M}_d \times \left(M_{t,+}^+ \backslash M_t^\perp\right) : \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t},\ w_t^s(\mathbb{Q}, w) \in M_{s,+}^+\Big\} \supseteq \mathcal{W}_t.$$

- ***Convex upper continuous (c.u.c.)***: for any $D \in \mathcal{G}(M_t; M_{t,-})$

$$R_t^{-1}(D) := \left\{X \in L_d^p(\mathcal{F}_T) : R_t(X) \cap D \neq \emptyset\right\}$$

is closed.

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  is closed.
- If $(R_t)_{t=0}^T$ is c.u.c. and convex then $\tilde{R}_t(\cdot) := R_t(-R_s(\cdot))$ is closed and convex.

Multi-Portfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized, c.u.c., and convex then the following are equivalent:

- $R_s(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_t(Y)$;
- For every $(\mathbb{Q}, w) \in \mathcal{W}_t$

$$-\beta_t^{\min}(\mathbb{Q}, w) = \operatorname{cl}\left(-\beta_{t,s}^{\min}(\mathbb{Q}, w) + \mathbb{E}_t^{\mathbb{Q}}\left[-\beta_s^{\min}(\mathbb{Q}, w_t^s(\mathbb{Q}, w))\right]\right).$$

Multi-Portfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized, c.u.c., and conditionally convex where

$$R_\tau(X) := \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_\tau^e} \left[-\alpha_\tau^{\min}(\mathbb{Q},w) + \left(\mathbb{E}_\tau^{\mathbb{Q}}\left[-X\right] + \Gamma_\tau(w)\right) \cap M_\tau\right]$$

then the following are equivalent:

- $R_s(X) \subseteq \bigcup_{Y\in\mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y\in\mathbf{Y}} R_t(Y)$;
- For every $(\mathbb{Q},w) \in \mathcal{W}_t^e$

$$-\alpha_t^{\min}(\mathbb{Q},w) = \operatorname{cl}\left(-\alpha_{t,s}^{\min}(\mathbb{Q},w) + \mathbb{E}_t^{\mathbb{Q}}\left[-\alpha_s^{\min}(\mathbb{Q},w_t^s(\mathbb{Q},w))\right]\right).$$

- Let $\mathbb{Q}, \mathbb{R} \in \mathcal{M}_d$ then $\mathbb{S} = \mathbb{Q} \oplus^s \mathbb{R}$ if $\frac{d\mathbb{S}}{d\mathbb{P}} := \xi_{0,s}(\mathbb{Q}) \cdot \xi_{s,T}(\mathbb{R})$ where

$$\xi^i_{t,s}(\mathbb{Q}) := \begin{cases} \mathbb{E}_s\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right] \Big/ \mathbb{E}_t\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right] & \text{on } \left\{\mathbb{E}_t\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right] > 0\right\} \\ 1 & \text{else} \end{cases}$$

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Stability

A set $W_t \subseteq \mathcal{W}_t$ is ***stable*** at time t with respect to $W_{t,s}$ and W_s if

- $(\mathbb{Q}, w) \in W_t$ implies $(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \in W_s$ and
- $(\mathbb{Q}, w) \in W_{t,s}$ and $\mathbb{R} \in \mathcal{M}_d$ such that $(\mathbb{R}, w_t^s(\mathbb{Q}, w)) \in W_s$ implies $(\mathbb{Q} \oplus^s \mathbb{R}, w) \in W_t$

Multi-Portfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized, c.u.c., and coherent then the following are equivalent:

- $R_s(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbf{Y}} R_t(Y)$;
- $\mathcal{W}_t^{\max}$ is stable with respect to $\mathcal{W}_{t,s}^{\max}$ and $\mathcal{W}_s^{\max}$;
- $\mathcal{W}_t^{\max} = \left\{(\mathbb{Q} \oplus^s \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,s}^{\max}, (\mathbb{R}, w_t^s(\mathbb{Q}, w)) \in \mathcal{W}_s^{\max}\right\}$;
- $\mathcal{W}_t^{\max} = \mathcal{W}_{t,s}^{\max} \cap H_t^s(\mathcal{W}_s^{\max})$ where $H_t^s(W) := \{(\mathbb{Q}, w) \in \mathcal{W}_t : (\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \in W\}$.

2.2 Time consistency: Composition of risk measures

Composition of One-Step Risk Measures

Let $(R_t)_{t=0}^T$ be a risk measure then $\left(\tilde{R}_t\right)_{t=0}^T$ is the multi-portfolio time consistent version if

$$\tilde{R}_T(X) := R_T(X); \qquad \tilde{R}_t(X) := \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z)$$

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- Also given by:

$$\begin{aligned}
\tilde{A}_t &:= A_{t,t+1} + \tilde{A}_{t+1}; \\
-\tilde{\beta}_t(\mathbb{Q}, w) &:= \operatorname{cl}\left(-\beta_{t,t+1}^{\min}(\mathbb{Q}, w) + \mathbb{E}_t^{\mathbb{Q}}\left[-\tilde{\beta}_{t+1}(\mathbb{Q}, w_t^{t+1}(\mathbb{Q}, w))\right]\right); \\
\widetilde{\mathcal{W}}_t &:= \mathcal{W}_{t,t+1}^{\max} \cap H_t^{t+1}(\widetilde{\mathcal{W}}_{t+1}); \\
-\tilde{\alpha}_t(\mathbb{Q}, w) &:= \operatorname{cl}\left(-\alpha_{t,t+1}^{\min}(\mathbb{Q}, w) + \mathbb{E}_t^{\mathbb{Q}}\left[-\tilde{\alpha}_{t+1}(\mathbb{Q}, w_t^{t+1}(\mathbb{Q}, w))\right]\right).
\end{aligned}$$

3.1 Examples: Superhedging

- Convex transaction costs at time t: closed convex set $\mathbb{R}^d_+ \subseteq K_t[\omega] \subseteq \mathbb{R}^d$ (solvency region), positions transferable into non-negative portfolios.

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Self-Financing Portfolio Process

A **self-financing portfolio process** $(V_t)_{t=0}^T$ is a stochastic process of portfolio vectors (of "physical units") if starting with no assets you can trade for V_t from the portfolio V_{t-1}.

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with $V_{-1} = 0$ and K_t is a solvency region.

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- Let $K_{t,s} := \sum_{r=t}^{s} L^p_d(\mathcal{F}_r; K_r)$ denote the portfolios reachable from time t at time s.

- $SHP_t(X) := \left\{u \in L^p_d(\mathcal{F}_t) : -X + u \in -K_{t,T}\right\}.$

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- If the market model $(K_t)_{t=0}^T$ satisfies proper no-arbitrage argument (robust no scalable arbitrage) then the superhedging portfolios can be found via the dual representation with penalty function:

$$-\alpha_t^{SHP}(\mathbb{Q}, w) = \sum_{s=t}^{T} \Big\{ u \in L^p_d(\mathcal{F}_t) : \\ \operatorname*{ess\,inf}_{k \in L^p_d(\mathcal{F}_s; K_s)} w^\mathsf{T} \mathbb{E}_t^{\mathbb{Q}}[k] \leq w^\mathsf{T} u \Big\}$$

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- This is multi-portfolio time consistent, but not necessarily self-recursive.

- If solvency cones (K_t is a.s. a cone) then the superhedging portfolios are conditionally coherent with dual variables defined by

$$\mathcal{W}_{\{t,...,T\}} = \{(\mathbb{Q}, w) \in \mathcal{W}_t : \forall s \in \{t, ..., T\} : \\ w_t^s(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_s; K_s^+)\}$$

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- This is self-recursive.

- Level $\lambda^t \in L_d^\infty(\mathcal{F}_t)$ bounded away from 0.
- Average Value-at-Risk at time t is closed and conditionally coherent risk measure defined by the dual variables:

$$\mathcal{W}_t^\lambda := \left\{ (\mathbb{Q}, w) \in \mathcal{W}_t : 0 \preceq w \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} \preceq w/\lambda^t \right\}$$

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- Average Value-at-Risk is not time consistent.

- Consider $M := \mathbb{R}^d$ and $p = +\infty$

3.2 Examples: Average Value-at-Risk

- Consider $M := \mathbb{R}^d$ and $p = +\infty$
- $AV@R_t$ is c.u.c.
- Multi-portfolio time consistent version has dual variables:

$$\begin{aligned}\widetilde{\mathcal{W}}_t^{\lambda} &:= \{(\mathbb{Q}, w) \in \mathcal{W}_t : \forall s \in \{t, t+1, ..., T-1\} : \\ &\qquad w_t^s(\mathbb{Q}, w)/\lambda^s \succeq w_t^{s+1}(\mathbb{Q}, w)\} \\ &= \{(\mathbb{Q}, w) \in \mathcal{W}_t : \forall s \in \{t, t+1, ..., T-1\} \forall i : \\ &\qquad \mathbb{P}\left(\mathbb{E}_{s+1}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right] \leq \frac{1}{\lambda_i^s}\mathbb{E}_s\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right] \text{ or } w_i = 0\right) = 1 \ \forall i\right\}\end{aligned}$$

3.3 Examples: Entropic risk measure

- Utility based shortfall risk measures:

$$R_t^u(X) := \{m \in M_t : \mathbb{E}_t\left[u(X+m)\right] \in C_t\}$$

for some vector utility function u and $C_t \in \mathcal{G}(L_d^p(\mathcal{F}_t); L_d^p(\mathcal{F}_t)_+)$

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- Restrictive entropic risk measure: $M = \mathbb{R}^d$, $u_i(x) = \frac{1-\exp(-\lambda_i x)}{\lambda_i}$ and $C_t = L_d^p(\mathcal{F}_t)_+$
- Closed convex risk measure with penalty function

$$-\alpha_t^{ent}(\mathbb{Q}, w) = H_t(\mathbb{Q}|\mathbb{P})/\lambda + \Gamma_t(w)$$
$$-\beta_t^{ent}(\mathbb{Q}, w) = H_t(\mathbb{Q}|\mathbb{P})/\lambda + G_t(w)$$

  where $\hat{H}_t(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_t^{\mathbb{Q}}\left[\log(\frac{d\mathbb{Q}}{d\mathbb{P}})\right]$
- c.u.c., normalized, and multi-portfolio time consistent.

Thank you

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