Distribution-Invariant Risk Measures, Information, and Dynamic Consistency

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November 24, 2003; this version March 11, 2005

Abstract

In the first part of the article, we characterize distribution-invariant risk measures with convex acceptance and rejection sets on the level of distributions. It is shown that these risk measures are closely related to utility-based shortfall risk.

In the second part of the paper, we provide an axiomatic characterization for distribution-invariant dynamic risk measures of terminal payments. We prove a representation theorem and investigate the relation to static risk measures. A key insight of the paper is that dynamic consistency and the notion of "measure convex sets of probability measures" are intimately related. This result implies that under weak conditions dynamically consistent dynamic risk measures can be represented by static utility-based shortfall risk.

Key words: Distribution-invariant risk measures, shortfall risk, utility functions, dynamic risk measure, capital requirement, measure of risk, dynamic consistency, measure convexity

JEL Classification: G18, G11, G28

Mathematics Subject Classification (2000): 91B16, 91B28, 91B30

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1 Introduction

Risk management and financial regulation relies on the proper assessment of the downside risk of financial positions. Traditional approaches – such as *value at risk* – do (in general) neither encourage diversification nor account for the size of extremely large losses. These drawbacks motivated an axiomatic analysis of risk measures with desirable properties. Robust representations of static risk measures are a consequence of convex analysis and have been considered in several articles, see e.g. Artzner, Delbaen, Eber & Heath (1999), Jaschke & Küchler (2001), Föllmer & Schied (2002*a*), Föllmer & Schied (2002*b*), Delbaen (2002) and Frittelli & Rosazza (2002). An excellent summary of these results can be found in Föllmer & Schied (2004).

While the temporal setting in all these approaches is static, realistic risk management and financial regulation requires *dynamic* risk measures for financial positions. When assessing the riskiness of cash flows as time evolves, two issues have to be taken into consideration: the risk measurements must consistently be updated, as new information becomes available, and intermediate cash flows must be taken into account, cf. Wang (1996), Wang (1999), Riedel (2004), Artzner, Delbaen, Eber, Heath & Ku (2003), Detlefsen (2003), Scandolo (2003), Cheridito, Delbaen & Kupper (2004*a*), Cheridito, Delbaen & Kupper (2004*b*), Cheridito, Delbaen & Kupper (2004*c*), and Föllmer & Penner (2004).

The current article contributes to both the theory of static and dynamic risk measures. In Sections 2 and 3 we investigate static distribution-invariant risk measures. Since these risk measures depend only on the distribution of financial positions, they can either be considered as functionals on spaces of random variables or spaces of distributions. In the first case, the convexity of the risk measures or, equivalently, their acceptance sets leads to refined robust representation theorems, cf. Kusuoka (2001), Carlier & Dana (2003), and Kunze (2003). In the current paper we consider the second perspective, i.e. the interpretation of risk measures as functionals of distributions. Convexity of acceptance and rejection sets on the level of laws of financial positions has a natural economic interpretation. If the acceptance set (resp. rejection set) is convex on the level of distributions, then any randomization over two acceptable (resp. rejected) positions is again acceptable (resp. rejected). Under additional topological conditions, we prove in Section 3 that risk measures with convex acceptance and rejection sets on the level of distributions coincide with utility-based shortfall risk. It turns out that this result is related to dynamic risk measures.

In Section 4 we propose a notion of distribution-invariant dynamic risk measures. We concentrate on the issue of updating, as new information becomes available; an extension to intermediate cash flows can be found in Weber (2004). The suggested dynamic risk measures can be represented by a vector of static distribution-invariant risk measures. The main focus of Section 4 is the issue of dynamic consistency. We propose two notions of dynamic consistency for dynamic risk measures, namely acceptance and rejection consistency. We call a dynamic risk measure acceptance consistent (resp. rejection consistent), if any position which is acceptable (resp. not acceptable) for sure in the future is already acceptable (resp. not acceptable) today. We show that a distribution-invariant risk measure which is both acceptance and rejection consistent can be represented by a *single* unique static distribution-invariant risk measure.

Conversely, we investigate when dynamic risk measures represented by a single static risk measure are acceptance and rejection consistent. Dynamic consistency is closely related to properties of the acceptance and rejection sets of the representing static risk measures. The concept of measure convex sets known from Choquet theory (see Winkler (1985)) leads to a complete characterization of the class of static risk measures that corresponds to consistent dynamic risk measures. In connection with the characterization theorem on static risk measures in Section 3 we can show that dynamically consistent, convex risk measures are represented by utility-based shortfall risk.

The paper is organized as follows. In Section 2 and 3 we investigate static distribution-invariant risk measures. Section 2 provides basic definitions and properties. In Section 3 we derive the main characterization theorem. Section 4 investigates distribution-invariant dynamic risk measures. Section 4.1 presents the simple representation in terms of static risk measures. Section 4.2 and 4.3 introduce the notions of dynamic consistency and investigate both its implications and connection to measure convexity. Section 4.4 relates these results to the characterization theorem of Section 3.

2 Static Risk Measures – Basic Definitions and Properties

Distribution-invariant risk measures are usually introduced as functionals on a space of random variables. They are characterized by three properties: inverse monotonicity, the translation property, and distribution-invariance; see e.g. Chapter 4 in Föllmer & Schied (2002*c*). For convenience, we recall the standard definition.

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be some probability space. A mapping $\Psi : L^\infty(\Omega) \to \mathbb{R}$ is called a distribution-invariant *classical risk measure* if it satisfies the following conditions for all $X, Y \in L^\infty(\Omega)$:

- *Inverse Monotonicity:* If $X \leq Y$, then $\Psi(X) \geq \Psi(Y)$.
- *Translation Property:* If $m \in \mathbb{R}$, then $\Psi(X + m) = \Psi(X) - m$.
- *Distribution-invariance:* If the distributions of X and Y under P are equal, then $\Psi(X) = \Psi(Y)$.

Monotonicity refers to the property that risk decreases if the payoff profile is increased. The translation property formalizes that risk is measured on a monetary scale: if a monetary amount $m \in \mathbb{R}$ is added to a position X, then the risk of X is reduced by m. Distribution-invariance refers to the fact that the functional takes the same value for random variables which have the same distribution; that is, the risk measure depends on the distribution only.

Due to distribution-invariance, any distribution-invariant risk measure (originally defined on a space of random variables) may also be interpreted as a functional on a space of distributions. This interpretation is very useful in the context of compound lotteries. For illustration, suppose $X, Y \in L^\infty$ are two financial positions with known risk. Now choose (independently of X and Y) position X with probability $\alpha \in [0, 1]$, and Y with probability $1 - \alpha$. It is a natural question to ask how the risk of the resulting compound lottery and the risks of X and Y are related to each other. An answer can be obtained in terms of functionals defined on a space of laws. For this reason we start with the following alternative definition of a risk measure.

Let $\mathcal{M}_{1,c}(\mathbb{R})$ be the space of probability measures on the real line with compact support. A partial order $\leq$ on $\mathcal{M}_{1,c}(\mathbb{R})$ is given by *stochastic dominance*, i.e. $\mu \leq \nu$ if $\mu(-\infty, x] \geq \nu(-\infty, x]$ for all $x \in \mathbb{R}$.

Definition 2.2. A mapping $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ is called a *risk measure* if it satisfies the following conditions for all $\mu, \nu \in \mathcal{M}_{1,c}(\mathbb{R})$:

- *Inverse Monotonicity:* If $\mu \leq \nu$, then $\Theta(\mu) \geq \Theta(\nu)$.
- *Translation Property:* If $m \in \mathbb{R}$, then $\Theta(\tilde{T}_m \, \mu) = \Theta(\mu) - m$.

  Here, for $m \in \mathbb{R}$ the translation operator $\tilde{T}_m$ is given by $(\tilde{T}_m \mu)(\cdot) = \mu(\cdot - m)$.

Remark 2.3. Definition 2.2 introduces risk measures as functionals on the space of probability measures on the real line, while the classical literature investigates functionals on spaces of random variables. Both concepts can easily be translated into each other by the following construction. Via this correspondence we will later derive properties of risk measures on $\mathcal{M}_{1,c}(\mathbb{R})$ from the classical case.

Suppose that $(\Omega', \mathcal{F}', P')$ is an atomless probability space. If $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ is a risk measure in the sense of Definition 2.2, then $\Psi(X) := \Theta(\mathcal{L}(X))$ defines a distribution-invariant risk measure Ψ on $L^\infty(\Omega', \mathcal{F}', P')$, cf. Definition 2.1. Here, $\mathcal{L}(X)$ designates the distribution of X under P. For convenience, we will denote by Θ' the risk measure Ψ induced by Θ.

Conversely, if Ψ is a distribution-invariant risk measure on $L^\infty(\Omega', \mathcal{F}', P')$ according to Definition 2.1, then $\Theta(\mu) := \Psi(X)$ for some $X \sim \mu$ defines a risk measure on $\mathcal{M}_{1,c}(\mathbb{R})$ in the sense of Definition 2.2. If $\Psi = \Theta'$, we recover by this procedure the original risk measure Θ.

As a basis for the analysis we need to investigate technical properties like continuity and measurability. Classical risk measures are Lipschitz continuous on L^∞. This translates to Vasserstein continuity of risk measures which are defined on spaces of distributions. An interpretation of this notion in terms of quantile functions is given in Remark 2.5 below. As a direct consequence we obtain measurability with respect to the standard σ-algebra on the space of distributions, cf. Corollary 2.6.

Lemma 2.4. *Any risk measure $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ is Lipschitz continuous with respect to the Vasserstein distance V_∞, that is $|\Theta(\mu) - \Theta(\nu)| \le V_\infty(\mu, \nu)$. Here, for $\mu, \nu \in \mathcal{M}_{1,c}(\mathbb{R})$ the Vasserstein distance is defined by $V_\infty(\mu, \nu) = \inf \|X - Y\|$, where $\|\cdot\|$ denotes the essential supremum and the infimum is taken over all pairs of random variables $X \sim \mu$ and $Y \sim \nu$ on some atomless probability space.*

Proof. The Lipschitz continuity is a simple consequence of Lemma 4.3 in Föllmer & Schied (2002*c*), see Weber (2004). □

Remark 2.5. The Vasserstein metric V_∞ can be represented in terms of the inverse of the distribution functions (i.e. the quantile functions) of the measures $\mu, \nu \in \mathcal{M}_{1,c}(\mathbb{R})$, cf. Owen (1987). We denote by F_μ^{-1} and F_ν^{-1} the right-continuous inverse of the distribution function of μ and ν, respectively. It holds that

$$V_\infty(\mu, \nu) = \sup_{0<u<1} |F_\mu^{-1}(u) - F_\nu^{-1}(u)|. \tag{2.1}$$

Corollary 2.6. *A risk measure $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ is measurable with respect to the Borel-σ-algebra on $\mathcal{M}_{1,c}(\mathbb{R})$ generated by the weak topology.*

Proof. The V_∞-metric generates the Borel-σ-algebra on $\mathcal{M}_{1,c}(\mathbb{R})$ induced by the weak topology, see e.g. Weber (2004). □

Associated with any risk measure are the sets of financial positions with strictly positive or non positive risk. Positions with non positive risk are usually interpreted as acceptable. On the contrary, positions with positive risk are interpreted as not acceptable. This leads to the following definition of acceptance and rejection sets.

Definition 2.7. Let Θ be a risk measure. Its *acceptance set* (resp. *rejection set*) on the level of probability distributions is defined by $\mathcal{N}_\Theta = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \ \Theta(\mu) \leq 0\}$ (resp. by $\mathcal{N}_\Theta^c$).

For any given risk measure, the acceptance set consists of all probability distributions with non positive risk; the rejection set is defined as its complement. Risk measures define uniquely their acceptance and rejection sets. Conversely, starting with a candidate acceptance set, a corresponding risk measure can be defined as a capital requirement, i.e. the minimal monetary amount that makes a position acceptable. The following lemma formalizes this idea and is obtained as an immediate corollary of the corresponding well-known result on classical risk measures, see e.g. Propositions 4.5 and 4.6 in Föllmer & Schied (2002*c*).

Lemma 2.8. *Assume that $\mathcal{N} \subseteq \mathcal{M}_{1,c}(\mathbb{R})$ is non-empty, and satisfies the following two conditions:*

$$\inf \{m \in \mathbb{R} : \quad \delta_m \in \mathcal{N}\} > -\infty. \tag{2.2}$$

$$\mu \in \mathcal{N}, \ \nu \in \mathcal{M}_{1,c}(\mathbb{R}), \ \nu \geq \mu \ \Rightarrow \ \nu \in \mathcal{N}. \tag{2.3}$$

Then $\mathcal{N}$ induces a risk measure Θ by

$$\Theta(\mu) = \inf\{m \in \mathbb{R} : \quad T_m(\mu) \in \mathcal{N}\}. \tag{2.4}$$

$\mathcal{N}$ is included in the acceptance set of Θ. If $\mathcal{N}$ is the acceptance set of a risk measure Θ, then Θ can be recovered from $\mathcal{N}$ by (2.4).

Finally, we translate the standard notion of "convexity" of risk measures to our setting. Convexity gives a precise meaning to the idea that diversification should not increase risk, see Section 4.1. in Föllmer & Schied (2002*c*). A classical risk measure $\Psi : L^\infty \to \mathbb{R}$ is *convex*, if

$$\Psi(\alpha X + (1-\alpha)Y) \leq \alpha\Psi(X) + (1-\alpha)\Psi(Y)$$

for all $X, Y \in L^\infty$, $\alpha \in [0,1]$.

In addition to convexity, we recall the following two concepts. Ψ is *positively homogenous*, if $\Psi(\lambda X) = \lambda\Psi(X)$ for all $X \in L^\infty$ and $\lambda \geq 0$. The risk measure is *coherent*, if it is both convex and positively homogenous.

Via their correspondence to classical risk measures, the notions of convexity and coherence are also well-defined for risk measures on $\mathcal{M}_{1,c}(\mathbb{R})$.

Definition 2.9. Let $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ and $\Theta' : L^\infty \to \mathbb{R}$ be risk measures as in Remark 2.3. We say that Θ is *convex* (resp. *coherent*) if Θ' is convex (resp. coherent).

Remark 2.10. Suppose an atomless probability space $(\Omega', \mathcal{F}', P')$ is given. Then a risk measure $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ defines a unique classical risk measure $\Theta' : \mathcal{L}^\infty(\Omega', \mathcal{F}', P') \to \mathbb{R}$, see Remark 2.3. It is not difficult to show that the notions of convexity and coherence do not depend on the choice of the atomless probability space $(\Omega', \mathcal{F}', P')$, see Weber (2004).

3 Static Risk Measures, Compound Lotteries, and Shortfall Risk

Distribution-invariant risk measure can either be interpreted as classical risk measures defined on a space of random variables or as functionals on a space of probability distributions. As we have seen in Remark 2.3, the two notion can easily be translated into each other. Using the notation introduced there, "convexity" refers to the convexity of $\Theta' : L^\infty \to \mathbb{R}$. In geometric terms, it can be restated as the convexity of the acceptance set on the level of random variables,

$$\{X \in L^\infty : \ \Theta'(X) \leq 0\}.$$

This type of convexity formalizes that a risk measure behaves sensible if positions are diversified.

A different type of convexity of risk measures is convexity on the level of distributions which can be formalized in terms of the functional $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$. Suppose that the risk measure's acceptance set on the level of probability distributions,

$$\mathcal{N}_\Theta = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \ \Theta(\mu) \leq 0\},$$

is convex. A natural interpretation of this property in terms of compound lotteries is the following. Whenever two probability measures μ and ν are acceptable and $\alpha \in [0,1]$ is some probability, then the compound lottery $\alpha\mu + (1-\alpha)\nu$, that randomizes over μ and ν, is also acceptable. Analogously, convexity of the rejection set, $\mathcal{N}_\Theta^c$, can be interpreted.

In this section we characterize risk measures with convex acceptance and rejection sets on the level of probability distributions. While these properties have a clear economic meaning in the context of the static risk measure, Section 4 demonstrates their relevance for the analysis of dynamic risk measures.

3.1 Technical Tools

We need to introduce weak topologies on $\mathcal{M}_{1,c}(\mathbb{R})$ that allow us to deal with integrals against unbounded test functions. For a fixed continuous function

$$\psi : \mathbb{R} \to [1, \infty)$$

we denote by C^ψ the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which we can find a constant $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$|f(x)| \leq c \cdot \psi(x).$$

ψ is called a *gauge function.* $\mathcal{M}_c^+(\mathbb{R})$ designates the space of finite measures with compact support.

Definition 3.1. The ψ-weak topology on the set $\mathcal{M}_c^+(\mathbb{R})$ is the initial topology of the family $\mu \mapsto \int f(x)\mu(dx)$ ($\mu \in \mathcal{M}_c^+(\mathbb{R})$, $f \in C^\psi$).

In other words, the ψ-weak topology is the weakest topology on $\mathcal{M}_c^+(\mathbb{R})$ for which all mappings $\mu \mapsto \int f(x)\mu(dx)$ ($\mu \in \mathcal{M}_c(\mathbb{R})$) with $f \in C^\psi$ are continuous. It is *finer* than the weak topology. Convergence of sequences of measures can be characterized as follows, see Corollary A.29 in Föllmer & Schied (2002*c*).

Lemma 3.2. *A sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_c^+(\mathbb{R})$ converges ψ-weakly to $\mu \in \mathcal{M}_c^+(\mathbb{R})$ if and only if*

$$\int f d\mu_n \longrightarrow \int f d\mu$$

for every measurable function f which is μ-almost everywhere continuous and for which exists a constant $c \in \mathbb{R}$ such that $|f| \leq c \cdot \psi$ μ-almost everywhere.

3.2 A Characterization Theorem

Under weak technical condition, risk measures with convex acceptance and rejection sets on the level of distributions can be characterized via a loss function. This implies that they are closely connected to von Neumann-Morgenstern utility theory and the particular risk measure *shortfall risk*, see Remark 3.4 and Example 3.8 below. The exact characterization is provided by the following theorem. Recall that a loss function is a non decreasing function which is not identically constant.

Theorem 3.3. *Let Θ be a risk measure on $\mathcal{M}_{1,c}(\mathbb{R})$, and let $\mathcal{N}$ be its acceptance set on the level of probability distributions. Assume that there exists $x \in \mathbb{R}$ with $\delta_x \in \mathcal{N}$ such that for $y \in \mathbb{R}$, $\delta_y \in \mathcal{N}^c$,*

$$(1-\alpha)\delta_x + \alpha\delta_y \in \mathcal{N} \tag{3.1}$$

for sufficiently small $\alpha > 0$. Then the following statements are equivalent:

(1) Both the acceptance set $\mathcal{N}$ and the rejection set $\mathcal{N}^c$ of Θ are convex, and $\mathcal{N}$ is ψ-weakly closed for some gauge function $\psi : \mathbb{R} \to [1, \infty)$.

(2) There exists a left-continuous loss function $\ell : \mathbb{R} \to \mathbb{R}$ and a scalar $z \in \mathbb{R}$ in the interior of the convex hull of the range of ℓ such that

$$\mathcal{N} = \left\{ \mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \int \ell(-x)\mu(dx) \leq z \right\}.$$

Remark 3.4. Before proving this theorem, let us emphasize that the risk measures characterized in Theorem 3.3 are closely connected to classical utility theory of von Neumann and Morgenstern.

Setting $u(x) := -\ell(-x)$, we can interpret u as a Bernoulli utility function. A financial position $\mu \in \mathcal{M}_{1,c}(\mathbb{R})$ is thus considered acceptable, if its expected utility is larger than $-z$,

$$U(\mu) := \int u(x)\mu(dx) \geq -z.$$

The remaining part of this section is devoted to two issues: the proof of the main theorem and a short investigation of continuity properties of the functional $\mu \mapsto \int \ell(-x)\mu(dx)$. A discussion of examples and consequences is deferred to Sections 3.3 and 3.4.

Proof of Theorem 3.3. (1) $\Rightarrow$ (2): The proof of this implication can be outlined as follows. In a first step we define an appropriate candidate loss function ℓ and a threshold level z. In a second step, a finite-dimensional separation argument allows us to verify that this function represents the acceptance set, if we focus on simple probability measures only. Third, we prove the left-continuity of ℓ. Finally, the general case can be derived by an approximation argument.

We choose $x_1 \in \mathbb{R}$ with $\delta_{x_1} \in \mathcal{N}$ according to (3.1), and let $x_2 \in \mathbb{R}$, $\delta_{x_2} \in \mathcal{N}^c$. These numbers are fixed for all parts of the proof.

a) Definition of the loss function ℓ and level z : For convenience, we define a function $g : \mathbb{R} \to \mathbb{R}$ with $\ell(-x) = g(x)$. We set

$$g(x_1) := 0, \tag{3.2}$$

$$g(x_2) := 1. \tag{3.3}$$

Let $z := \sup\{0 \leq \alpha \leq 1 : \alpha\delta_{x_2} + (1-\alpha)\delta_{x_1} \in \mathcal{N}\}$. Since $\mathcal{N}$ is ψ-weakly closed, the supremum is actually a maximum. Thus, $z \neq 1$, since $\delta_{x_2} \notin \mathcal{N}$. By (3.1) $z > 0$, hence $z \in (0,1)$. Hence, z is in the interior of the convex hull of the range of g.

Since $\mathcal{N}$ is ψ-weakly closed, it follows from inverse monotonicity that there exists a threshold level $r \in \mathbb{R}$ such that $[r,\infty) = \{y \in \mathbb{R} : \delta_y \in \mathcal{N}\}$, $(-\infty, r) = \{y \in \mathbb{R} : \delta_y \in \mathcal{N}^c\}$.

If $y \in [r, \infty)$, define

$$\bar{\alpha}(y) := \sup\{0 \le \alpha \le 1 : \alpha\delta_{x_2} + (1-\alpha)\delta_y \in \mathcal{N}\}. \tag{3.4}$$

Since $\mathcal{N}$ is ψ-weakly closed, the supremum is actually a maximum. Thus $\bar{\alpha}(y) \neq 1$, since $\delta_{x_2} \notin \mathcal{N}$. Hence, $1 - \bar{\alpha}(y) \neq 0$, and we may define

$$g(y) := \frac{z - \bar{\alpha}(y)}{1 - \bar{\alpha}(y)}. \tag{3.5}$$

Inverse monotonicity implies additionally that $y \mapsto \bar{\alpha}(y)$ is increasing on $[r, \infty)$. Hence, $y \mapsto g(y) = 1 + \frac{z-1}{1-\bar{\alpha}(y)}$ is decreasing on $[r, \infty)$, since $z - 1 < 0$.

If $y \in (-\infty, r)$, define

$$\bar{\alpha}(y) := \sup\{0 \le \alpha \le 1 : \alpha\delta_y + (1-\alpha)\delta_{x_1} \in \mathcal{N}\}. \tag{3.6}$$

Observe that $\bar{\alpha}(y) \neq 1$, since $\delta_y \notin \mathcal{N}$. By (3.1) we have $\bar{\alpha}(y) \neq 0$. We let

$$g(y) := \frac{z}{\bar{\alpha}(y)}. \tag{3.7}$$

Inverse monotonicity implies that $y \mapsto \bar{\alpha}(y)$ is increasing on $(-\infty, r)$. Hence $y \mapsto g(y)$ is decreasing on $(-\infty, r)$.

Moreover, note that on the one hand $g(y) > z$ for $y \in (-\infty, r)$. On the other hand, $g(y) = z + (z-1)\frac{\bar{\alpha}(y)}{1-\bar{\alpha}(y)} \le z$ for $y \in [r, \infty)$, since $z - 1 < 0$. Hence, $g : \mathbb{R} \to \mathbb{R}$ is a decreasing function, thus ℓ increasing.

b) Simple probability measures: For probability measures of the form $\mu = \sum_{i=1}^n \beta_i \cdot \delta_{x_i}$ with $\beta_i \ge 0$, $x_i \in \mathbb{R}$ $(i = 3, \ldots, n)$, $\sum_{i=1}^n \beta_i = 1$, $n \in \mathbb{N}$, and x_1, x_2 as chosen above, we will show that

$$\mu \in \mathcal{N} \Leftrightarrow \int g(x)\mu(dx) \le z.$$

Let $\mu = \sum_{i=1}^n \beta_i \cdot \delta_{x_i}$ be given. We denote by $\mathcal{M}$ the convex hull of $\{\delta_{x_i} : \ i = 1, 2, \ldots, n\}$. The simplex $\mathcal{M}$ is a convex subset of the n-dimensional vector space spanned by $\{\delta_{x_i} : \ i = 1, 2, \ldots, n\}$. Let $\mathcal{A} := \mathcal{N} \cap \mathcal{M}$, $\mathcal{B} = \mathcal{N}^c \cap \mathcal{M}$. Then $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$, the sets $\mathcal{A}$ and $\mathcal{B}$ are both convex, and $\mathcal{A}$ is closed in the Euclidian topology. We can therefore find an affine functional $h : \mathcal{M} \to \mathbb{R}$ and $q \in \mathbb{R}$ such that

$$\begin{aligned} h(\mu) &\leq q, \qquad \mu \in \mathcal{A}, \\ h(\mu) &> q, \qquad \mu \in \mathcal{B}. \end{aligned}$$

We define a function k by

$$k := \frac{h - h(\delta_{x_1})}{h(\delta_{x_2}) - h(\delta_{x_1})}.$$

Then

$$\begin{aligned} k(\mu) &\leq \frac{q - h(\delta_{x_1})}{h(\delta_{x_2}) - h(\delta_{x_1})}, \qquad \mu \in \mathcal{A}, \\ k(\mu) &> \frac{q - h(\delta_{x_1})}{h(\delta_{x_2}) - h(\delta_{x_1})}, \qquad \mu \in \mathcal{B}. \end{aligned}$$

We show now that $g(x_i) = k(\delta_{x_i})$. For $i = 1, 2$ the claim is immediate from the definition of k. This implies for $\alpha \in (0, 1)$ that

$$k(\alpha\delta_{x_2} + (1 - \alpha)\delta_{x_1}) = \alpha.$$

Hence,

$$\begin{aligned} z &= \sup\{0 \leq \alpha \leq 1 : \ \alpha\delta_{x_2} + (1 - \alpha)\delta_{x_1} \in \mathcal{N}\} \\ &= \sup\left\{0 \leq \alpha \leq 1 : \ \alpha \leq \frac{q - h(\delta_{x_1})}{h(\delta_{x_2}) - h(\delta_{x_1})}\right\} = \frac{q - h(\delta_{x_1})}{h(\delta_{x_2}) - h(\delta_{x_1})}. \end{aligned}$$

Let now $i \neq 1, 2$, and $\bar{\alpha} : \mathbb{R} \to \mathbb{R}$ as defined in (3.4) and (3.6). Assume first that $x_i \in [r, \infty)$ with r as chosen in part a). This implies that

$$\begin{aligned} \bar{\alpha}(x_i) &= \sup\{0 \leq \alpha \leq 1 : \ \alpha\delta_{x_2} + (1 - \alpha)\delta_{x_i} \in \mathcal{N}\} \\ &= \sup\{0 \leq \alpha \leq 1 : \ \alpha + (1 - \alpha)k(\delta_{x_i}) \leq z\}. \end{aligned}$$

Observe that $\bar{\alpha}(x_i) \neq 1$ and that $\alpha \mapsto \alpha + (1 - \alpha)k(\delta_{x_i})$ is continuous. Hence, the last equation is satisfied, if and only if $\bar{\alpha}(x_i) + (1 - \bar{\alpha}(x_i))k(\delta_{x_i}) = z$, i.e.

$$k(\delta_{x_i}) = \frac{z - \bar{\alpha}(x_i)}{1 - \bar{\alpha}(x_i)} = g(x_i).$$

Second, consider the case $x_i \in (-\infty, r)$. Then

$$\bar{\alpha}(x_i) \;=\; \sup\{0 \le \alpha \le 1: \; \alpha\delta_{x_i} + (1-\alpha)\delta_{x_1} \in \mathcal{N}\}$$

$$=\; \sup\{0 \le \alpha \le 1: \;\; \alpha k(\delta_{x_i}) \;\le\; z\}.$$

Observe that $\bar{\alpha}(x_i) \neq 1$ and that $\alpha \mapsto \alpha k(\delta_{x_i})$ is continuous. Hence, the last equation is satisfied, if and only if $\bar{\alpha}(x_i)k(\delta_{x_i}) = z$, i.e.

$$k(\delta_{x_i}) = \frac{z}{\bar{\alpha}(x_i)} = g(x_i).$$

Finally, we obtain for $\mu = \sum_{i=1}^n \beta_i \delta_{x_i}$ that

$$\mu \in \mathcal{N} \;\Leftrightarrow\; k(\mu) \le z \;\Leftrightarrow\; \sum_{i=1}^n \beta_i g(x_i) \le z \;\Leftrightarrow\; \int g(x)\mu(dx) \le z.$$

c) Left-continuity of ℓ: Next we prove that g is right-continuous, thus ℓ left-continuous. Since g is decreasing, $g(x+)$ exists for each $x \in \mathbb{R}$. We have already shown that $g(x_1) < z$, $g(x_2) > z$. This implies that for given $x \in \mathbb{R}$ we can find $\alpha \in (0,1]$ and $w \in \mathbb{R}$ such that

$$\alpha g(x+) + (1-\alpha)g(w) = z.$$

Let $x_n \searrow x$. Since g is decreasing, we obtain $\alpha\delta_{x_n} + (1-\alpha)\delta_w \in \mathcal{N}$ ($n \in \mathbb{N}$). Moreover, $\alpha\delta_{x_n} + (1-\alpha)\delta_w$ converges ψ-weakly to $\alpha\delta_x + (1-\alpha)\delta_w$. It follows that $\alpha\delta_x + (1-\alpha)\delta_w \in \mathcal{N}$, since $\mathcal{N}$ is ψ-weakly closed. Thus,

$$z \;\ge\; \alpha g(x) + (1-\alpha)g(w) \;\ge\; \alpha g(x+) + (1-\alpha)g(w) \;=\; z.$$

Therefore, $g(x) = g(x+)$.

d) General probability measures: Finally, we will show that the representation of $\mathcal{N}$ via the function g is not restricted to simple probability measures. First, let $\mu \in \mathcal{N}$. There exists a decreasing sequence of simple probability measures $(\mu_n)_n \subseteq \mathcal{M}_{1,c}(\mathbb{R})$ converging to μ ψ-weakly from above. By inverse monotonicity, $(\mu_n)_n \subseteq \mathcal{N}$, thus $z \;\ge\; \int g(x)\mu_n(dx)$. Letting $(Y_n)_n$ resp. Y be the quantile functions of the measures $(\mu_n)_n$ resp. μ, we obtain that $Y_n \searrow Y$ Lebesgue-a.e. By the right-continuity of g,

$g(Y_n) \to g(Y)$ Lebesgue-a.e. Thus, by Lebesgue's dominated convergence theorem,

$$\int g(x)\mu_n(dx) = \int_0^1 g(Y_n(s))ds \longrightarrow \int_0^1 g(Y(s))ds = \int g(x)\mu(dx).$$

Conversely, let $z \geq \int g(x)\mu(dx)$. Then there exists a decreasing sequence of simple probability measures $(\mu_n)_n \subseteq \mathcal{M}_{1,c}(\mathbb{R})$ converging ψ-weakly to μ from above. Since g is decreasing, we obtain $z \geq \int g(x)\mu_n(dx)$, thus $(\mu_n)_n \subseteq \mathcal{N}$. Since $\mathcal{N}$ is ψ-weakly closed, we obtain $\mu \in \mathcal{N}$.

(2) $\Rightarrow$ (1): The convexity of the acceptance and rejection sets of Θ is immediate. We need to show that the acceptance set is ψ-weakly closed for some suitable gauge function ψ.

Let $\psi \in C(\mathbb{R})$, $\psi \geq |g| + 1$ with $g(x) = \ell(-x)$ $(x \in \mathbb{R})$. We show that the functional $\mu \mapsto \int g(x)\mu(dx)$ is lower semicontinuous with respect to the ψ-weak topology. Since the ψ-weak topology on $\mathcal{M}_{1,c}(\mathbb{R})$ is metrizable, we employ the sequential characterization of closed sets. Let $z \in \mathbb{R}$ be given, and let $(\mu_n)_n \subseteq \mathcal{M}_{1,c}(\mathbb{R})$, $\mu_n \to \mu \in \mathcal{M}_{1,c}(\mathbb{R})$ ψ-weakly, where $\int g(x)\mu_n(dx) \leq z$ for $n \in \mathbb{N}$.

By Skorohod coupling we can find bounded random variables $(X_n)_n$, X on some probability space $(\Omega, \mathcal{F}, P)$ such that $X_n \sim \mu_n$ $(n \in \mathbb{N})$, $X \sim \mu$, $X_n \to X$ P-a.s.

We have $\lim \psi(X_n) = \psi(X)$ P-almost surely, and $\lim \int \psi(X_n)dP = \int \psi(X)dP$. Observe that $\psi(X_n) + g(X_n) \geq 0$ $(n \in \mathbb{N})$. By Fatou's Lemma we obtain that

$$\begin{aligned} \int \psi(X)dP + z &\geq \int \psi(X)dP + \liminf_n \int g(X_n)dP \\ = \quad \liminf_n \int (\psi(X_n) + g(X_n))dP &\geq \int \liminf_n (\psi(X_n) + g(X_n))dP \\ = \quad \int \psi(X)dP + \int \liminf_n g(X_n)dP &\geq \int \psi(X)dP + \int g(X)dP. \end{aligned}$$

The last inequality follows from the fact that g is decreasing and right-continuous, since $X_n \to X$ P-almost surely. Hence, $z \geq \int g(X)dP = \int g(x)\mu(dx)$. □

The functional $\mu \mapsto \int \ell(-x)\mu(dx)$ together with the threshold level z defines the acceptance set in Theorem 3.3(2). Let us finally discuss the topological properties of this functional.

Remark 3.5. The acceptance set $\mathcal{N}$ is ψ-weakly closed for some gauge function ψ. Thus, any position which is approximated by acceptable positions in this topology is again acceptable. However, the functional $\mu \mapsto \int \ell(-x)\mu(dx)$ might possibly not be ψ-weakly continuous for any gauge function ψ. To be more precise, it is ψ-weakly continuous for some gauge function ψ, if and only if ℓ is continuous. This follows from the representation of the dual space of $\mathcal{M}_{1,c}(\mathbb{R})$ endowed with the ψ-weak topology, cf. Lemma 3.6.

Instead of continuity, a weaker property is always satisfied, namely lower semicontinuity. Letting $\psi \in C(\mathbb{R})$, $\psi \geq |g| + 1$ with $g(x) = \ell(-x)$ $(x \in \mathbb{R})$, the functional $\mu \mapsto \int \ell(-x)\mu(dx)$ is lower semicontinuous for the ψ-weak topology (see the proof of Theorem 3.3).

Lemma 3.6. *Let $I : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ be an affine, ψ-weakly continuous functional. Then there exists $g \in C^\psi$ such that*

$$I(\mu) = \int g(x)\mu(dx) \qquad\qquad (\mu \in \mathcal{M}_{1,c}(\mathbb{R})).$$

Proof. The result is immediate from duality theory, see Chapter VIII in Werner (2002); alternatively, a direct construction can be found in Weber (2004). □

3.3 Examples

Theorem 3.3 characterizes the class of risk measures whose acceptance set is defined in terms of a loss function. A necessary and sufficient condition can be phrased in terms of compound lotteries: compound lotteries of acceptable positions are acceptable, while compound lotteries of rejected positions are rejected. The additional topological condition is weak and requires only that positions which can be approximated in a rather fine topology are again acceptable. The overall assumption (3.1) requires that for some (large) monetary amount, any loss is acceptable, as long as its probability is sufficiently small.

Example 3.7. While property (3.1) is satisfied by many examples, it is not shared by the worst case measure. Modulo null sets, the worst case measure is the least upper bound for the potential loss which can occur for any potential outcome. To be more precise, condition (3.1) excludes that Θ

equals the worst case measure plus some constant (say r), i.e.

$$\Theta(\mu) \;=\; r \;-\; \text{ess inf}\,\mu \qquad\qquad (\mu \in \mathcal{M}_{1,c}(\mathbb{R})).$$

In contrast the following risk measures satisfy the conditions of Theorem 3.3 .

Example 3.8. (1) For the *negative expected value*, $\Theta(\mu) = -\int x\mu(dx)$, the loss function is given by $\ell(x) = x$ with threshold $z = 0$.

(2) For *value at risk* at level $\lambda \in (0,1)$, defined as

$$\begin{aligned}\mathrm{VaR}_\lambda(\mu) &= -\inf\{y \in \mathbb{R} : \mu(-\infty, y] > \lambda\}\\ &= -\sup\{y \in \mathbb{R} : \mu(-\infty, y) \le \lambda\}\\ &= \inf\{y \in \mathbb{R} : \mu(-\infty, -y) \le \lambda\},\end{aligned}$$

the loss function equals $\ell(x) = \mathbf{1}_{(0,\infty)}$ with threshold $z = \lambda$.

(3) If the loss function ℓ is convex, the associated convex risk measure is called (utility-based) *shortfall risk*. Then ℓ is continuous and the level z belongs to the interior of the range of ℓ. An exponential loss function $\ell(x) = \exp(ax)$, $a > 0$, leads to the special case of the *entropic risk measure*

$$\Theta(\mu) = \frac{1}{a}\left(\log \int \exp(-ax)\mu(dx) - \log z\right).$$

Another risk measure is *average value at risk* which appears in the literature also under the names *worst conditional expectation*, *conditional value at risk*, and *expected shortfall*. It does not satisfy the hypotheses of Theorem 3.3.

Example 3.9. For a given level $x \in (0,1)$, let VaR_x be value at risk at level x as defined in Example 3.8. For $\lambda \in (0,1)$ *average value at risk* at level λ is defined by

$$\mathrm{AVaR}_\lambda(\mu) = \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_x(\mu)dx, \quad \mu \in \mathcal{M}_{1,c}(\mathbb{R}).$$

As the next example will show, the acceptance set of AVaR_λ ($\lambda \in (0,1)$) is not a convex subset of the space of probability measures. Hence, AVaR_λ does *not* satisfy condition (1) of Theorem 3.3, and its acceptance set cannot be represented in terms of a loss function.

Example 3.10. The acceptance set of AVaR_λ ($\lambda \in (0,1)$) is not a convex subset of the space of probability measures. For each $\lambda \in (0,1)$ this can be demonstrated by the following counterexample.

We let $\mu = \lambda \cdot \mathrm{unif}\,[-1,1] \;+\; (1-\lambda)\cdot \mathrm{unif}\,[1,2]$, $\nu = \delta_0$. Then we obtain for the quantile function of μ that

$$q_\mu(\gamma) = \frac{2\gamma}{\lambda} - 1, \qquad\qquad (\gamma \le \lambda).$$

Hence, $\mathrm{AVaR}_\lambda(\mu) = 0$. Moreover, $\mathrm{AVaR}_\lambda(\nu) = 0$. This implies $\mu, \nu \in \mathcal{N}$. Let $\alpha = \frac{\lambda}{2-\lambda}$. Then $q_{\alpha\nu+(1-\alpha)\mu}(\lambda) = 0$. But

$$q_{\alpha\nu+(1-\alpha)\mu}(\gamma) = \begin{cases} \frac{2\gamma}{(1-\alpha)\lambda} - 1 \;<\; 0 & \text{if } \gamma < \frac{(1-\alpha)\lambda}{2} \\ 0 & \text{if } \frac{(1-\alpha)\lambda}{2} \le \gamma \le \lambda \end{cases}$$

Hence, $\mathrm{AVaR}_\lambda(\alpha\nu + (1-\alpha)\mu) > 0$. This implies that $\alpha\nu + (1-\alpha)\mu \notin \mathcal{N}$. The acceptance set of AVaR_λ is therefore not a convex subset of the space of probability measures.

3.4 Convex and Coherent Risk Measures

The following corollary connects the preceding characterization theorem with the classical theory of convex risk measures, cf. Chapter 4.6. in Föllmer & Schied (2002*c*).

Corollary 3.11. *Let $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ be a risk measure, and assume that its acceptance set $\mathcal{N}$ satisfies condition (2) of Theorem 3.3. Then Θ is convex, if and only if the loss function ℓ is convex.*

Proof. If ℓ is convex, the corresponding risk measure is clearly convex. We only have to prove the other direction. Assume thus that ℓ is not convex. Set $g(x) = \ell(-x)$. Then g is not convex, and we can find $x, y \in \mathbb{R}$, $x < y$, such that

$$\frac{g(x) + g(y)}{2} \;<\; g\left(\frac{x+y}{2}\right).$$

Because z is in the interior of the convex hull of the range of g, we can always find $w \in \mathbb{R}$ and $\alpha \in [0,1)$ such that

$$\alpha g(w) \;+\; (1-\alpha)\cdot\left(\frac{g(x)+g(y)}{2}\right) \;\le\; z \;<\; \alpha g(w) \;+\; (1-\alpha)\cdot g\left(\frac{x+y}{2}\right).$$

We define the following random variables on $(0,1)$ with Lebesgue measure λ:

$$Z_1 \;=\; w\cdot\mathbf{1}_{(0,\alpha)} \;+\; x\cdot\mathbf{1}_{[\alpha,(1+\alpha)/2)} \;+\; y\cdot\mathbf{1}_{[(1+\alpha)/2,1)}$$

$$Z_2 \;=\; w\cdot\mathbf{1}_{(0,\alpha)} \;+\; y\cdot\mathbf{1}_{[\alpha,(1+\alpha)/2)} \;+\; x\cdot\mathbf{1}_{[(1+\alpha)/2,1)}$$

Then Z_1 and Z_2 are both acceptable, since for $i=1,2$,

$$\int g(Z_i)d\lambda \;=\; \alpha g(w) \;+\; (1-\alpha)\left(\frac{g(x)+g(y)}{2}\right) \;\le\; z.$$

We define $Z := \frac{Z_1+Z_2}{2} = w\cdot\mathbf{1}_{(0,\alpha)} + \frac{x+y}{2}\cdot\mathbf{1}_{[\alpha,1)}$, and obtain

$$\int g(Z)d\lambda \;=\; \alpha g(w) \;+\; (1-\alpha)\cdot g\left(\frac{x+y}{2}\right) \;>\; z.$$

Hence, Z is not acceptable, contradicting the convexity of Θ. $\square$

Theorem 3.3 and Corollary 3.11 imply that any convex risk measure Θ on $\mathcal{M}_{1,c}(\mathbb{R})$ with convex acceptance and rejection set can be represented as *shortfall risk*, if the acceptance set is ψ-weakly closed for some gauge function. Shortfall risk allows a robust representation in terms of the Fenchel-Legendre transform of the associated loss function.

Lemma 3.12. *Let Θ be shortfall risk associated with a convex and continuous loss function ℓ. We denote the Fenchel-Legendre transform of ℓ by*

$$\ell^*(y) := \sup_{x\in\mathbb{R}}(yx - \ell(x)).$$

A robust representation of the risk measure is given by

$$\Theta(\mu) = \max_{\nu\in\mathcal{M}_1(\mu)}\left(-\int x\nu(dx) - \alpha(\nu|\mu)\right) \qquad (\mu \in \mathcal{M}_{1,c}(\mathbb{R})).$$

Here, $\mathcal{M}_1(\mu)$ is the set of probability measures which are absolutely continuous with respect to μ. The penalty function α is given by

$$\alpha(\nu|\mu) = \inf_{\lambda>0} \frac{1}{\lambda} \left(z + \int \ell^* \left(\lambda \frac{d\nu}{d\mu} \right) d\mu \right) \qquad (\nu \in \mathcal{M}_1(\mu)).$$

Proof. We apply Theorem 4.61 of Föllmer & Schied (2002c). For $\mu \in \mathcal{M}_{1,c}(\mathbb{R})$, let $P := \mu$ and $X := id$. By $\mathcal{X}$ we denote the class of all bounded measurable functions. Of course, $(\mathbb{R}, \mathcal{B}, P)$ is not necessarily atomless. Nevertheless, if $\mathcal{L}(Y)$ $(Y \in \mathcal{X})$ denotes the distribution of Y under P, then $Y \mapsto \Theta(\mathcal{L}(Y))$ $(Y \in \mathcal{X})$ defines a convex risk measure on $\mathcal{X}$ which satisfies the conditions of Proposition 4.59 and Theorem 4.61 of Föllmer & Schied (2002c). This implies Lemma 3.12. □

Example 3.13. In the case of the entropic risk measure with loss function $\ell(x) = \exp(a \cdot x)$, $a > 0$, a penalty function can be defined in terms of the relative entropy:

$$\alpha(\nu|\mu) = \frac{1}{a} \left(H(\nu|\mu) - \log z \right) \qquad (\nu \in \mathcal{M}_1(\mu)).$$

Here, the relative entropy is given by

$$H(\nu|\mu) = \begin{cases} \int \frac{d\nu}{d\mu} \log\left(\frac{d\nu}{d\mu}\right) d\mu & \text{if } \nu \ll \mu, \\ \infty & \text{else.} \end{cases}$$

Example 3.14. Another example that allows explicit calculations is given by the convex loss functional

$$\ell(x) = \begin{cases} \frac{1}{p}\, x^p & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $p > 1$ (see e.g. Föllmer & Schied (2002c), Example 4.64). Denoting by $q = p/(p-1)$ the dual coefficient, the Legendre-Fenchel transform is calculated as

$$\ell^*(y) = \begin{cases} \frac{1}{q}\, y^q & \text{if } y \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

A penalty function is then given by

$$\alpha^p(\nu|\mu) = (p \cdot z)^{1/p} \left(\int \left(\frac{d\nu}{d\mu} \right)^q d\mu \right)^{1/q} \qquad (\nu \in \mathcal{M}_1(\mu)).$$

The case of classical expected shortfall risk $\ell(x) = x^+$ is obtained for $p \searrow 1$. A penalty function can be calculated as

$$\alpha(\nu|\mu) \;=\; z \cdot \left\| \frac{d\nu}{d\mu} \right\| \qquad\qquad (\nu \in \mathcal{M}_1(\mu)).$$

Finally we consider the case of coherent risk measures. In this case, the convex loss function ℓ is piecewise linear with a kink at 0.

Corollary 3.15. *Let $\Theta : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ be a risk measure, and assume that its acceptance set $\mathcal{N}$ satisfies condition (2) of Theorem 3.3. Then Θ is coherent, if and only if $\ell(x) = z + \alpha x^+ - \beta x^-$ for $\alpha \geq \beta > 0$.*

Proof. First, let $\ell(x) = z + \alpha x^+ - \beta x^-$ be given. Since $\alpha \geq \beta > 0$, the loss function ℓ is convex. Hence, ℓ induces a convex risk measure. Let $\mu \in \mathcal{N}$, and let $X \sim \mu$ be a random variable on some atomless probability space $(\Omega, \mathcal{F}, P)$. Then for $\lambda \geq 0$,

$$\int \ell(-\lambda X)dP = z + \lambda \int (\ell(-X) - z)dP = (1-\lambda)z + \lambda \int \ell(-X)dP \leq z.$$

This implies that $\mathcal{L}(\lambda X) \in \mathcal{N}$. Hence, Θ is positively homogeneous.

Conversely, let Θ be a coherent risk measure that satisfies the hypotheses. Then Θ can be represented by a continuous and convex loss function ℓ and a threshold level $z \in \mathbb{R}$ in the interior of the range of ℓ. Since Θ is positively homogeneous, $\delta_y \in \mathcal{N}$ for $y \in [0, \infty)$ and $\delta_y \in \mathcal{N}^c$ for $y \in (-\infty, 0)$. This implies that $\ell(0) = z$. Subtracting z, we may w.l.o.g. assume that $z = 0$ and $\ell(0) = 0$. Let $g(x) := \ell(-x)$.

Suppose that there exist $x' \in \mathbb{R}$, $\lambda' \geq 0$ such that $g(\lambda' x') \neq \lambda' g(x')$. Since g is convex and $g(0) = 0$, this implies that there exist $x \in \mathbb{R}$ and $\lambda > 1$ such that $g(\lambda x) > \lambda g(x)$. Since $z = 0$ lies in the interior of the range of g, we can find $w_1, w_2 \in \mathbb{R}$ such that $g(w_1) < 0 < g(w_2)$. Therefore there exist $w \in \mathbb{R}$ and $\alpha \in (0, 1]$ such that

$$\alpha g(x) + (1-\alpha)g(w) = 0.$$

Hence, $\alpha\delta_x + (1-\alpha)\delta_w \in \mathcal{N}$. Since g is convex with $g(0) = 0$, $g(\lambda w) \geq \lambda g(w)$. Since $g(\lambda x) > \lambda g(x)$, we obtain

$$\alpha g(\lambda x) + (1-\alpha) g(\lambda w) > 0.$$

This implies that $\alpha\delta_{\lambda x} + (1-\alpha)\delta_{\lambda w} \notin \mathcal{N}$ – contradicting the assumption of coherence. Altogether we obtain that for $x \in \mathbb{R}$, $\lambda \geq 0$ it holds that $\lambda g(x) = g(\lambda x)$. This implies that g is of the form

$$g(x) = \alpha x^- - \beta x^+$$

for $\alpha, \beta \in \mathbb{R}$. $\alpha, \beta \geq 0$, since g is decreasing. The inequality $\alpha \geq \beta$ follows from the convexity of g. Finally, $\alpha, \beta > 0$, because 0 lies in the interior of the range of g. □

For coherent measures of risk that satisfy the assumptions of Theorem 3.3 a position is acceptable, if a suitable weighted average of expected gains and expected losses is sufficiently large. Gains and losses can be weighted differently, but the weight on the losses must not be smaller than the weight on the gains.

The class of coherent shortfall risk measures is characterized by the three parameters (z, α, β). The parameter z which equals both the threshold level and $\ell(0)$ is not essential, since modifying it does not change the risk measure. Thus, coherent shortfall risk measures form a two parameter family.

4 Dynamic Risk Measures

Consider time steps $t = 0, 1, \ldots, T$ with $T \geq 2$ and a standard Borel probability space $(\Omega, \mathcal{F}, P)$. T is a fixed time horizon, at which all payments of financial positions are made. $(\mathcal{F}_t)_{t=0,1,\ldots,T}$ denotes a filtration. We assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. By P_t^T we denote the price of a zero-coupon bond at time $t = 0, 1, .., T-1$ with face value 1 maturing at final time T. To be more precise: we assume that P_t^T, $t = 0, 1, .., T-1$, is a $\mathcal{F}_t$-measurable random variable which is bounded in $[\epsilon, c]$ for some $0 < \epsilon < c$.

Definition 4.1. For $t = 0, 1, ..., T-1$ let $\rho_t : L^\infty(\Omega, \mathcal{F}, P) \times \Omega \to \mathbb{R}$ be a functional. The sequence $\rho := (\rho_t)_{t=0,1,...,T-1}$ is called a *dynamic risk measure* if the following axioms are satisfied for $D, D' \in L^\infty(\Omega, \mathcal{F}, P)$:

(A1) *Adaptedness and Boundedness*:

$\rho_t(D) \in L^\infty(\Omega, \mathcal{F}_t, P)$

(A2) *Inverse Monotonicity*:

If $D \geq D'$ P-almost surely, then $\rho_t(D) \leq \rho_t(D')$ P -almost surely.

(A3) *Translation Property*:

If $Z \in L^\infty(\Omega, \mathcal{F}_t, P)$, then P-a.s.

$$\rho_t\left(D + \frac{Z}{P_t^T}\right) = \rho_t(D) - Z.$$

Remark 4.2. $D \in L^\infty(\Omega, \mathcal{F}, P)$ is interpreted as a cash flow at the terminal date T. A1 ensures that the risk $\rho_t(D)$ of the position D evaluated at time t depends only on information available at time t (adaptedness). Since the position D is bounded, it is reasonable that its risk is also bounded. A2 states that the downside risk of a position decreases, if the payoff of the position increases P-a.s.

The translation property, A3, formalizes the idea that $\rho_t(D)$ is a capital requirement. If an investor invests an amount of Z at time t in a risk-free way until maturity T, her risk is reduced exactly by Z. In particular, A3 implies that P-a.s.

$$\rho_t\left(D + \frac{\rho_t(D)}{P_t^T}\right) = 0.$$

We will interpret $\rho_t(D)$ as the monetary amount that should be added to D at time t and invested in risk-free bonds until the final date to make the position acceptable from the point of view of an investor or regulator, given the information at time t. A position D is acceptable at time t, iff its risk $\rho_t(D) \leq 0$. In this case, no positive monetary amount has to be added to the position to make it acceptable.

Definition 4.3. Let ρ be a dynamic risk measure. The *acceptance indicator* $a = (a_t)_{t=0,1,\ldots,T-1}$ of ρ is defined by

$$a_t(D) := \mathbf{1}_{\{\rho_t(D) \leq 0\}}.$$

Clearly, the acceptance indicator $a_t(D)(\omega)$ equals 1, iff D is acceptable at time t in scenario $\omega \in \Omega$. Otherwise, it takes the value 0. The acceptance indicator allows us to introduce the notion of *distribution-invariant* dynamic risk measures.

We denote by $\mathcal{M}_{1,c}(\mathbb{R})$ the space of *probability measures on the real line* with compact support. If Y is a measurable function defined on $(\Omega, \mathcal{F}, P)$ into any standard Borel space $(S, \mathcal{B})$, we denote by $\mathcal{L}(Y|\mathcal{F}_t)$ a *regular conditional distribution* of Y given $\mathcal{F}_t$, see Section 10.2 in Dudley (2002). Letting $\mathcal{M}_1(S)$ be the space of probability measures on S endowed with the σ-algebra induced by the mappings $\pi_B : \mu \mapsto \mu(B)$ $(B \in \mathcal{B})$, the mapping $\Omega \to \mathcal{M}_1(S)$, $\omega \mapsto \mathcal{L}(Y|\mathcal{F}_t)(\omega)$ is measurable, see Chapter 1 in Kallenberg (1997).

Definition 4.4. The dynamic risk measure ρ is called *distribution-invariant*, if there exists a measurable mapping $H_t : \mathcal{M}_{1,c}(\mathbb{R}) \to \{0, 1\}$ such that for all $D \in L^\infty$,

$$a_t(D) = H_t(\mathcal{L}(D_T|\mathcal{F}_t)) \quad P\text{-almost surely.}$$

Remark 4.5. Distribution-invariance of dynamic risk measures formalizes the idea that whether or not a financial position is acceptable at date t depends only on its conditional distribution.

4.1 A Simple Representation Theorem

The following representation characterizes distribution-invariant dynamic risk measures in terms of static risk measures. It states that any distribution-invariant dynamic risk measure can be represented by a vector of static risk measures.

Theorem 4.6. *Assume that the filtered probability space is rich in the sense that there exists a* $\text{unif}(0,1)$*-distributed random variable independent of* $\mathcal{F}_{T-1}$*. Then any distribution-invariant dynamic* *risk measure ρ can be represented by a sequence $(\Theta_t)_{t=0,1,\ldots,T-1}$ of static risk measures $\Theta_t : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$, $t = 0, 1, \ldots T-1$, i.e. for all $D \in L^\infty(\Omega, \mathcal{F}, P)$*

$$\rho_t(D) = P_t^T \cdot \Theta_t \left(\mathcal{L}(D|\mathcal{F}_t)\right) \qquad P\text{-almost surely.} \tag{4.1}$$

The risk measures Θ_t in the representation are unique, and the acceptance set of Θ_t is given by

$$\mathcal{N}_t = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \; H_t(\mu) = 1\}. \tag{4.2}$$

$H_t : \mathcal{M}_{1,c} \to \{0, 1\}$ is the mapping introduced in Definition 4.4.

Before proving this theorem, let us state that also the converse statement holds.

Lemma 4.7. *Let $(\Theta_t)_{t=0,1,\ldots,T-1}$ be a sequence of static risk measures as introduced in Definition 2.2. Then (4.1) defines a distribution-invariant dynamic risk measure.*

The remainder of this section is devoted to the proofs of Theorem 4.6 and Lemma 4.7 which link static and dynamic risk measures.

Proof of Theorem 4.6. For $t = 0, 1, \ldots, T-1$ we define the sets $\mathcal{N}_t = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \; H_t(\mu) = 1\}$. We show that $\mathcal{N}_t$ induces a static risk measure. First, we prove that property (2.2) holds: Letting $M' \in L^\infty(\Omega, \mathcal{F}, P)$ be arbitrary, we define

$$M := M' + \frac{\rho_t(M') - 1}{P_t^T}.$$

By assumption, P_t^T is bounded away from zero and $\rho_t(M') \in L^\infty(\Omega, \mathcal{F}, P)$. Thus, $M \in L^\infty(\Omega, \mathcal{F}, P)$. By the translation property,

$$\rho_t(M) = \rho_t\left(M' + \frac{\rho_t(M') - 1}{P_t^T}\right) = \rho_t(M') - \rho_t(M') + 1 > 0.$$

Let $m \in \mathbb{R}$, $m \leq -\|M\|_\infty$. By inverse monotonicity, $\rho_t(m) \geq \rho_t(M) > 0$. Hence,

$$H_t(\delta_m) = a_t(m) = 0.$$

This implies that $\inf\{m \in \mathbb{R} : \; \delta_m \in \mathcal{N}_t\} > -\infty$.

Second, we prove property (2.3): Let $\mu \in \mathcal{N}_t$, $\nu \in \mathcal{M}_{1,c}(\mathbb{R})$, and $\nu \geq \mu$. Since the filtered probability space is rich, there exists a random variable Z uniformly distributed on $(0,1)$ and independent of $\mathcal{F}_{T-1}$. Define $M := q_\mu(Z) \sim \mu$ and $N := q_\nu(Z) \sim \nu$, where q_μ and q_ν are the quantile functions of μ and ν, respectively. Since ν stochastically dominates μ, we have $N \geq M$. By monotonicity, $\rho_t(N) \leq \rho_t(M)$. This implies $H_t(\nu) = 1$, since $H_t(\mu) = 1$ by assumption. Hence, $\nu \in \mathcal{N}_t$.

We denote the static risk measure induced by the set $\mathcal{N}_t$ by Θ_t and have to show that for given $D \in L^\infty(\Omega, \mathcal{F}, P)$ P-almost surely

$$\rho_t(D) = P_t^T \cdot \Theta_t(\mathcal{L}(D|\mathcal{F}_t)).$$

By $\tilde{T} : \mathbb{R} \times \mathcal{M}_{1,c}(\mathbb{R}) \to \mathcal{M}_{1,c}(\mathbb{R})$ we denote the translation operator, i.e. $\tilde{T}_r\mu(A) = \mu(A-r)$ for $r \in \mathbb{R}$, $\mu \in \mathcal{M}_{1,c}(\mathbb{R})$ and measurable $A \subseteq \mathbb{R}$.

Since $\rho_t(D) \cdot \left(P_t^T\right)^{-1}$ is $\mathcal{F}_t$-measurable and bounded, we get P-almost surely

$$\frac{\rho_t(D)}{P_t^T} = \text{ess inf } \left\{ m \in L^\infty(\Omega, \mathcal{F}_t, P) : \ \frac{\rho_t(D)}{P_t^T} \leq m \ P - \text{almost surely} \right\}$$

Now let $m \in L^\infty(\Omega, \mathcal{F}_t, P)$ be arbitrary. By the translation property, $\frac{\rho_t(D)}{P_t^T} - m = \frac{\rho_t(D+m)}{P_t^T}$. Thus, P-almost surely

$$\frac{\rho_t(D)}{P_t^T} \leq m \ \Leftrightarrow \ \rho_t(D+m) \leq 0 \ \Leftrightarrow \ \mathcal{L}(D+m|\mathcal{F}_t) \in \mathcal{N}_t \ \Leftrightarrow \ \tilde{T}_m\mathcal{L}(D|\mathcal{F}_t) \in \mathcal{N}_t.$$

This implies that P-a.s.

$$\frac{\rho_t(D)}{P_t^T} = \text{ess inf } \left\{ m \in L^\infty(\Omega, \mathcal{F}_t, P) : \ \ \tilde{T}_{m(\omega)}\mathcal{L}(D|\mathcal{F}_t)(\omega) \in \mathcal{N}_t \text{ for } P\text{−}a.a.\ \omega \in \Omega \right\}$$

We have to show that the right hand side equals P-a.s. $\Theta_t(\mathcal{L}(D|\mathcal{F}_t))$:

First, observe that $\Theta_t : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ is Lipschitz continuous with respect to the Vasserstein metric V_∞. This implies that $\hat{m} := \Theta_t(\mathcal{L}(D|\mathcal{F}_t)) \in L^\infty(\Omega, \mathcal{F}_t, P)$. Clearly, $\tilde{T}_{\hat{m}(\omega)}\mathcal{L}(D|\mathcal{F}_t)(\omega) \in \mathcal{N}_t$ for P-a.a. $\omega \in \Omega$. Thus, $\hat{m} \geq \frac{\rho_t(D)}{P_t^T}$ P-a.s.

Second, let $m \in L^\infty(\Omega, \mathcal{F}_t, P)$ such that $\tilde{T}_{m(\omega)}\mathcal{L}(D|\mathcal{F}_t)(\omega) \in \mathcal{N}_t$ for P-a.a. $\omega \in \Omega$. Since

$$\hat{m}(\omega) = \Theta_t(\mathcal{L}(D|\mathcal{F}_t)(\omega)) = \inf\{r \in \mathbb{R} : \ \tilde{T}_r\mathcal{L}(D|\mathcal{F}_t)(\omega) \in \mathcal{N}_t\},$$

we obtain in particular $\hat{m}(\omega) \leq m(\omega)$ for P-a.a. $\omega \in \Omega$. Hence $\hat{m} \leq \frac{\rho_t(D)}{P_t^T}$ P-a.s.

Finally, we show that $\mathcal{N}_t$ is indeed the acceptance set of Θ_t and that the representation is unique. Since the filtered probability space is rich, for μ we can find $M \in L^\infty$ with $\mathcal{L}(M|\mathcal{F}_t) = \mu$ P-a.s. Uniqueness is implied by the equality

$$\Theta_t(\mu) = \frac{\rho_t(M)}{P_t^T} \qquad P\text{-almost surely.}$$

Moreover, if $\Theta_t(\mu) \leq 0$, then $H_t(\mu) = 1$, thus $\mu \in \mathcal{N}_t$. This implies that $\mathcal{N}_t$ is indeed the acceptance set of Θ_t. □

Proof of Lemma 4.7. Adaptedness and inverse monotonicity are immediate. Boundedness follows from the boundedness assumptions on the bond prices and the Lipschitz continuity of static risk measures with respect to the Vasserstein metric V_∞.

The translation property can be verified as follows. Denote by $\tilde{T} : \mathbb{R} \times \mathcal{M}_{1,c}(\mathbb{R}) \to \mathcal{M}_{1,c}(\mathbb{R})$ the translation operator, and let $D \in L^\infty(\Omega, \mathcal{F}, P)$, $Z \in L^\infty(\Omega, \mathcal{F}_t, P)$. Then P-almost surely

$$\begin{aligned} \rho_t\left(D + \frac{Z}{P_t^T}\right) &= P_t^T \cdot \Theta_t\left(\mathcal{L}\left(D + \frac{Z}{P_t^T}\middle|\mathcal{F}_t\right)\right) = P_t^T \cdot \Theta_t\left(\tilde{T}_{\frac{Z}{P_t^T}}\ \mathcal{L}\left(D|\,\mathcal{F}_t\right)\right) \\ &= P_t^T \cdot \Theta_t\left(\mathcal{L}\left(D|\,\mathcal{F}_t\right)\right) - Z \;=\; \rho_t(D) - Z \end{aligned}$$

□

4.2 Dynamic Consistency

Theorem 4.6 shows that any distribution-invariant risk measure can be represented by a vector of static risk measures. Since every such vector defines a proper dynamic risk measure according to Lemma 4.7, risk measurements at different points in time need not be related to each other. From an economic point of view this is certainly problematic and some consistency across time seems appropriate. In the current section, we will suggest two notions of time consistency of dynamic risk measures. In the remaining part of the article, we are going to discuss the consequences of these requirements. It turns out that these are closely connected to convexity properties of the representing static risk measures.

Definition 4.8. A distribution-invariant dynamic risk measure ρ is

- *acceptance consistent*, if for all $t = 0, 1, \ldots, T-2$ and all $D \in L^\infty(\Omega, \mathcal{F}, P)$,

$$a_{t+1}(D) = 1 \quad P\text{-almost surely} \qquad \Longrightarrow \qquad a_t(D) = 1 \quad P\text{-almost surely;}$$

- *rejection consistent*, if for all $t = 0, 1, \ldots, T-2$ and all $D \in L^\infty(\Omega, \mathcal{F}, P)$,

$$a_{t+1}(D) = 0 \quad P\text{-almost surely} \qquad \Longrightarrow \qquad a_t(D) = 0 \quad P\text{-almost surely.}$$

Remark 4.9. *Acceptance consistency* captures the following intuition. If a position D is acceptable at the date $t+1$ irrespectively of actual scenario $\omega \in \Omega$ (modulo nullsets), then D should also be accepted at the earlier time t. In other words: "Why should we reject a position today, if we accept it tomorrow anyway?" *Rejection consistency* states the idea that a position should already be rejected at time t, if it is rejected at the later date $t+1$ for P-almost all scenarios $\omega \in \Omega$: "Why should we accept a position today, if we reject it tomorrow anyway?"

These dynamic consistency conditions have strong implications for the representation of a distribution-invariant dynamic risk measure in terms of static risk measures. In particular, if a dynamic risk measure is both acceptance and rejection consistent, all components of the vector of representing static risk measures are equal. Thus, the dynamic risk measure is represented by a single static risk measure. This fact is stated in the following corollary.

Corollary 4.10. *Assume that the filtered probability space is rich in the sense that there exists a* $\mathrm{unif}(0,1)$*-distributed random variable independent of* $\mathcal{F}_{T-1}$*. Let* ρ *be a distribution-invariant dynamic risk measure, and let* $\Theta_t : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ *be the representing static risk measures with acceptance sets* $\mathcal{N}_t$*,* $t = 0, 1, \ldots T-1$.

(1) *If ρ is acceptance consistent, then $\mathcal{N}_{t+1} \subseteq \mathcal{N}_t$, $t = 0, 1, \dots T-2$.*

(2) *If ρ is rejection consistent, then $\mathcal{N}_{t+1} \supseteq \mathcal{N}_t$, $t = 0, 1, \dots T-2$.*

(3) *If ρ is both acceptance and rejection consistent, then $\Theta_0 = \Theta_t$ for all $t = 1, 2, \dots, T-1$, and ρ can be represented by the single static risk measure $\Theta := \Theta_0$, i.e. for $D \in L^\infty(\Omega, \mathcal{F}, P)$,*

$$\rho_t(D) = P_t^T \cdot \Theta\left(\mathcal{L}(D|\mathcal{F}_t)\right) \qquad P\text{-almost surely.} \tag{4.3}$$

Proof. Assume that ρ is acceptance consistent. Let $\mu \in \mathcal{N}_{t+1}$. Since the probability space is rich, there exists a random variable $Z \sim \text{unif}(0,1)$ independent of $\mathcal{F}_{T-1}$. We define $D = q_\mu(Z)$ where q_μ is the quantile function of μ. Observe that $\mathcal{L}(D|\mathcal{F}_t) = \mathcal{L}(D|\mathcal{F}_{t+1}) = \mu$. We obtain that

$$1 = H_{t+1}(\mu) = a_{t+1}(D) = a_t(D) = H_t(\mu).$$

Hence, $\mu \in \mathcal{N}_t$. If ρ is rejection consistent, the proof is analogous. □

4.3 Consistency and Mixtures of Distributions

According to Corollary 4.10 a dynamic risk measure can be represented by one universal static risk measure, if it is both acceptance and rejection consistent. Conversely, one may ask: under which conditions is a distribution-invariant risk measure of the form (4.3) acceptance or rejection consistent? Necessary and sufficient conditions are closely connected to the convexity of acceptance and rejection sets on the level of distributions.

Definition 4.11. Let $\mathcal{C}$ be a measurable subset of $\mathcal{M}_{1,c}(\mathbb{R})$. We say that $\mathcal{C}$ is *locally measure convex* if for all $c \in \mathbb{R}$ and any probability measure γ on $\mathcal{C} \cap \mathcal{M}_1([-c, c])$ the mixture $\int \nu\gamma(d\nu)$ is again an element of $\mathcal{C}$ where $\mathcal{M}_1([-c, c])$ denotes the set of probability measures supported in the interval $[-c, c]$.

The last definition provides a localization of the notion of measure convex sets (see Winkler (1985)) which is appropriate in the context of probability measures with *compact* support.

Theorem 4.12. *Let Θ be a static risk measure, $\mathcal{N} \subseteq \mathcal{M}_{1,c}(\mathbb{R})$ be its acceptance set, and ρ be the distribution-invariant dynamic risk measure defined by*

$$\rho_t(D) = P_t^T \cdot \Theta\left(\mathcal{L}(D|\mathcal{F}_t)\right) \qquad P\text{-almost surely.} \tag{4.4}$$

If $\mathcal{N}$ is locally measure convex, then ρ is acceptance consistent. If $\mathcal{N}^c$ is locally measure convex, then ρ is rejection consistent.

Proof. We prove that ρ is acceptance consistent, if $\mathcal{N}$ is locally measure convex. The case of rejection consistency can be derived similarly.

Let $t \in \{0, 1, \ldots, T-2\}$, $D \in L^\infty(\Omega, \mathcal{F}, P)$, and $c \in \mathbb{R}$ such that $D \in [-c, c]$ P-almost surely. Define now a kernel Q_t from $(\Omega, \mathcal{F}_t)$ to $(\Omega, \mathcal{F})$ such that for measurable $A \subseteq \Omega$,

$$Q_t(\omega, A) = P(A|\mathcal{F}_t)(\omega)$$

Set $\mu_s := \mathcal{L}(D|\mathcal{F}_s)$ $(s = t, t+1)$. Then we obtain by disintegration for P-almost every $\omega \in \Omega$ that

$$\mu_t(\omega, \cdot) = \int \mu_{t+1}(\omega', \cdot) Q_t(\omega, d\omega')$$

Suppose that $a_{t+1}(D) = 1$ P-a.s. Then $\mu_{t+1}(\omega', \cdot) \in \mathcal{N} \cap \mathcal{M}_1([-c, c])$ for P-almost all $\omega' \in \Omega$. Hence for P-almost all $\omega \in \Omega$,

$$\mu_t(\omega, \cdot) = \int \mu_{t+1}(\omega', \cdot) Q_t(\omega, d\omega') \in \mathcal{N},$$

since $\mathcal{N}$ is locally measure convex. This implies $a_t(D) = 1$ P-a.s. Therefore, ρ is acceptance consistent.

□

The characterization of consistency in terms of the acceptance sets of the representing risk measure and mixtures of probability measures can be strengthened if the underlying filtered probability space is rich enough.

Definition 4.13. The filtered probability space is called *sequentially rich* if there exist both a $\text{unif}(0,1)$-distributed random variable independent of $\mathcal{F}_{T-1}$, and a $\text{unif}(0,1)$-distributed, $\mathcal{F}_{T-1}-$measurable random variable independent of $\mathcal{F}_{T-2}$.

Theorem 4.14. *Let the underlying probability space be sequentially rich, and assume that the dynamic risk measure ρ is represented as in Theorem 4.12.*

Then ρ is acceptance consistent, if and only if $\mathcal{N}$ is locally measure convex. Analogously, ρ is rejection consistent, if and only if $\mathcal{N}^c$ is locally measure convex.

Proof. We have already proven one direction in Theorem 4.12. Thus, we only need to show that 'consistency' implies 'measure convexity'. We will focus on the case of acceptance consistency. The case of rejection consistency works analogously.

Let ρ be a distribution-invariant dynamic risk measure, and let $\mathcal{N}$ be the corresponding acceptance set of the representing static risk measure Θ. Observe that $\mathcal{N}$ is measurable by definition of the functions H_t. Let $c \in \mathbb{R}$ be given, and let γ be a probability measure on $\mathcal{N} \cap \mathcal{M}([-c, c])$. Let $Z \sim \text{unif}(0,1)$ be a random variable independent of $\mathcal{F}_{T-1}$, and let $U \sim \text{unif}(0,1)$ be a $\mathcal{F}_{T-1}$- measurable random variable independent of $\mathcal{F}_{T-2}$. By Borel's theorem (see Theorem 2.19 in Kallenberg (1997)) there exists a measurable function $\mu : [0,1] \to \mathcal{N}$ such that $\mu(U) \sim \gamma$. We define a kernel from $\mathcal{M}_1(\mathbb{R})$ to $\mathbb{R}$ by

$$\begin{cases} \mathcal{M}_1(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) & \to & [0,1] \\ (\nu, A) & \mapsto & \nu(A) \end{cases}$$

By the kernel randomization lemma (see Lemma 2.22 in Kallenberg (1997)) there exists a measurable function

$$q : \mathcal{M}_1(\mathbb{R}) \times [0,1] \to \mathbb{R}$$

such that $q_\nu(Z) = q(\nu, Z) \sim \nu$. Clearly, the composite function $q_{\mu(\cdot)}(\cdot) : [0,1]^2 \to \mathbb{R}$ is measurable. We define the random variable $D := q_{\mu(U)}(Z) \in [-c, c]$. We obtain that for P-almost all $\omega \in \Omega$,

$$\mathcal{L}(D|\mathcal{F}_{T-1})(\omega) = \mu(U(\omega)) \in \mathcal{N}, \tag{4.5}$$

$$\mathcal{L}(D|\mathcal{F}_{T-2}) = \mathcal{L}(D) = \int_{\mathcal{N}} \nu\gamma(d\nu). \tag{4.6}$$

Equation (4.5) implies $a_{T-1}(D) = 1$ P-a.s. From acceptance consistence follows $a_{T-2}(D) = 1$ P-a.s. Thus, $\int_{\mathcal{N}} \nu\gamma(d\nu) \overset{(4.6)}{=} \mathcal{L}(D|\mathcal{F}_{T-2}) \in \mathcal{N}$. □

4.4 Dynamic Risk Measures and Shortfall Risk

The results of the Section 3 can be applied to dynamic risk measures. Recall that under weak additional conditions any static risk measure which is distribution-invariant and convex can be represented by shortfall risk, if both acceptance and rejection set are convex on the level of distributions. For the precise statement we refer to Theorem 3.3 and its Corollary 3.11. Dynamic consistency, convexity and a weak closure property imply that a dynamic risk measure can be represented in terms of shortfall risk.

Theorem 4.15. *Assume that the filtered probability space is sequentially rich. Let ρ be a distribution-invariant dynamic risk measure. We make the following assumptions:*

(1) ρ is acceptance and rejection consistent.

(2) ρ is convex in the sense that for $t = 0, 1, \dots, T-1$, $\alpha \in (0,1)$, $D, G \in \mathcal{D}$,

$$\rho_t(\alpha D + (1-\alpha)G) \leq \alpha \rho_t(D) + (1-\alpha)\rho_t(G).$$

(3) The set $\mathcal{N} = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \ H_t(\mu) = 1\}$ $(t = 0, 1, \dots, T-1)$ is ψ-weakly closed for some gauge function $\psi : \mathbb{R} \to [1, \infty)$.

(4) Assume that there exists $x \in \mathbb{R}$ with $\delta_x \in \mathcal{N}$ such that for $y \in \mathbb{R}$, $\delta_y \in \mathcal{N}^c$,

$$(1-\alpha)\delta_x + \alpha\delta_y \in \mathcal{N}$$

for sufficiently small $\alpha > 0$.

Then there exists a convex loss function $\ell : \mathbb{R} \to \mathbb{R}$ with associated shortfall risk measure Θ on $\mathcal{M}_{1,c}(\mathbb{R})$ such that ρ can be represented by Θ, i.e. for $D \in L^\infty(\Omega, \mathcal{F}, P)$,

$$\rho_t(D) = P_t^T \cdot \Theta\left(\mathcal{L}(D|\mathcal{F}_t)\right) \qquad P\text{-almost surely} \tag{4.7}$$

Proof. By Corollary 4.10 there exists a unique risk measure Θ such that ρ can be represented according to (4.7). By Theorem 4.14 the acceptance set $\mathcal{N}$ and the rejection set $\mathcal{N}^c$ are locally measure convex, thus convex. Hence, $\mathcal{N}$ can be represented according to Theorem 3.3 for some loss function $\ell : \mathbb{R} \to \mathbb{R}$. The convexity of ρ implies the convexity of Θ. It follows by Corollary 3.11 that ℓ is convex and therefore continuous. Hence, Θ is the shortfall risk measure associated with the continuous and convex loss function ℓ. $\square$

Remark 4.16. If a dynamic risk measure is distribution-invariant and satisfies at the same time conditions (1) – (4), then it can be represented by utility-based shortfall risk. Let us finally discuss these assumptions.

Distribution-invariance refers to the fact that risk measurements rely on the conditional distributions of financial positions only and not on the specific mechanism that generates them. In particular, dependence with other economic variables is neglected.

Assumption (1) specifies that risk measurements at different points in time should not contradict each other, see Definition 4.8 and the subsequent remark. Assumption (2) formalizes the idea that diversification should not increase risk.

The interpretation of assumption (3) and (4) is closely related to assumption (1). Due to time consistency, the dynamic risk measure ρ can be represented by a single static risk measure Θ with acceptance set $\mathcal{N}$ on the level of distributions. This allows us to rephrase assumptions (3) and (4) in terms of the initial date $t = 0$.

Assumption (3) is equivalent to the following condition: if the distribution of a position $D \in \mathcal{D}$ can be approximated in a rather fine topology by a sequence of positions acceptable at time 0, then D is also acceptable at time 0. Finally, assumption (4) requires that at time 0 any loss is acceptable for a position paying some large monetary amount otherwise, as long as the loss probability is sufficiently small.

From the point of view of an investor or regulator, distribution-invariance, convexity, and dynamic consistency might be desirable properties of a dynamic risk measure. The additional requirement on $\mathcal{N}$ to be ψ-weakly closed for some gauge function ψ is very weak and is even economically meaningful: terminal positions which can be approximated by acceptable positions in a rather fine topology are again acceptable. Also, requiring a tradeoff of large losses against arbitrarily small probabilities does not seem to be a very strong assumption. We argue therefore that static shortfall risk provides a good basis for the dynamic evaluation of financial positions.

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