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Tail expansions for random record distributions

By RUDOLF GRÜBEL and NIKLAS VON ÖHSEN
Institut für Mathematische Stochastik, Universität Hannover
Postfach 60 09, *D*-30060 *Hannover, Germany*
e-mail: rgrubel@stochastik.uni_hannover.de

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Abstract

The random record distribution ν associated with a probability distribution μ can be written as a convolution series, $\nu = \sum_{n=1}^{\infty} n^{-1}(n+1)^{-1}\mu^{\star n}$. Various authors have obtained results on the behaviour of the tails $\nu((x,\infty))$ as $x \to \infty$, using Laplace transforms and the associated Abelian and Tauberian theorems. Here we use Gelfand transforms and the Wiener–Lévy–Gelfand Theorem to obtain expansions of the tails under moment conditions on μ. The results differ notably from those known for other convolution series.

1. *Introduction and main result*

Random record models have been introduced by Gaver [**8**], who gave several applications. For the variant to be discussed in the present paper let X_n, $n \in \mathbb{N}$, be independent random variables with distribution μ. The corresponding partial sums $S_0 := 0$, $S_n := \sum_{m=1}^{n} X_m$ for $n \in \mathbb{N}$, constitute a renewal process with lifetime distribution μ. This classical terminology refers to the case where the X-variables are non-negative, in the general case we call $(S_n)_{n\in\mathbb{N}_0}$ a random walk with step distribution μ. Let $(Y_n)_{n\in\mathbb{N}_0}$ be another sequence of independent and identically distributed random variables, with continuous distribution function F_Y and independent of $(S_n)_{n\in\mathbb{N}_0}$. We regard Y_n as a label attached to S_n. The index τ of the first record of the Y-sequence is the infimum of all $n \in \mathbb{N}$ with the property that $Y_n > Y_m$ for $m = 0, \ldots, n-1$ and S_τ is the first *random record*. Strictly monotone transformations of the Y-values do not change τ, so the distribution ν_{rr} of S_τ depends on μ only and not on F. We call ν_{rr} the *random record distribution* associated with μ, writing $\nu_{\mathrm{rr}}(\mu)$ if we want to emphasize the dependence on μ.

With '$\star$' denoting convolution so that $\mu^{\star n}$ is the distribution of S_n and using the well-known elementary fact that $P(\tau = n) = n^{-1}(n+1)^{-1}$ for all $n \in \mathbb{N}$ (see e.g. [**7**, section I·5]), we obtain the basic representation

$$\nu_{\mathrm{rr}} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\,\mu^{\star n} \tag{1·1}$$

of the random record distribution as a convolution series in μ. The first moment of τ is obviously infinite, which implies that, as a rule, the expected value of S_τ does not exist (we exclude the trivial case where μ is concentrated at 0). This naturally raises the question of the behaviour of the tails of S_τ, i.e. the asymptotics of $\nu_{\rm rr}((x,\infty))$ as $x \to \infty$. The first results in this direction were obtained by Westcott [**21**, **22**], who considered the one-sided case and gave conditions that implied $\nu_{\rm rr}((x,\infty)) \sim m_1/x$, where $m_1 = m_1(\mu) := \int x\mu(dx)$ denotes the first moment associated with μ. Westcott also considered heavy-tailed distributions, with e.g. $\mu((x,\infty))$ of regular variation with index $-\alpha$, $0 < \alpha < 1$, and showed that in such cases the ratio of the $\nu_{\rm rr}$- and μ-tail behaves like $\alpha \log(x)$; Embrechts and Omey [**4**] who obtained a converse to this statement. The proofs in these papers are based on Abel–Tauber results for Laplace transforms (see [**2**] for a standard reference in this area).

Convolution series such as (1·1) appear in a variety of situations and special cases are of considerable importance in applied probability theory. With all coefficients equal to one we obtain the renewal measure $\nu_{\rm ren} := \sum_{n=1}^\infty \mu^{\star n}$, a central object of renewal theory. The series $\nu_{\rm harm} := \sum_{n=1}^\infty n^{-1}\mu^{\star n}$ is the harmonic renewal measure, which plays an important role in connection with the Wiener–Hopf factors of a distribution, in turn very useful for the analysis of random walks via ladder variables (see [**10**, **14**, **15**] and the references given therein). Generally, if the coefficients are non-negative and sum to 1, then the series arises as the distribution of a random sum. This includes (1·1). Cases in which the coefficients are the probabilities of a Poisson or geometric distribution are important in insurance mathematics and risk theory, where they are used in connection with total claim size distributions and ruin probabilities. The Poisson case is also essential to the analysis of infinitely divisible distributions. [**7**] continues to be a central reference for much of this material, but see also [**1**].

In all these cases the tail behaviour of the resulting ν or of normalized versions thereof has been investigated thoroughly. For renewal measures this is related to the rate of convergence in the renewal theorem (see e.g. [**13**] and the references given therein). A typical result is the following,

$$\nu_{\rm ren}((-\infty,x]) = c_1 \cdot x + c_2 + c_3 \int_x^\infty \int_y^\infty \mu((z,\infty))\,dz\,dy + r(x) \tag{1·2}$$

as $x \to \infty$ with some lower order error term $r(x)$, where the coefficients c_1, c_2, c_3 depend on μ via its moments. For harmonic renewal measures see [**10**] for the one-sided case and [**12**] for the general case. Once more, we cite a typical result:

$$\nu_{\rm harm}((-\infty,x]) = \log x + c_1 + c_2 \int_x^\infty \mu((y,\infty))\,dy + r(x), \tag{1·3}$$

with c_1, c_2 again depending on μ via its moments. For compound distributions with rapidly decreasing probability coefficients see again [**13**] and the references given therein: if e.g. $\mu((x,\infty))$ is regularly varying then

$$\nu((x,\infty)) \sim c_1\mu((x,\infty)), \tag{1·4}$$

i.e. the compound distribution tail is asymptotically equal to a multiple of the tail of the input measure.

On a technique-of-proof level, a general picture that seems to have emerged from these efforts is that the classical Abel–Tauber approach via Laplace transforms can often be complemented (in the sense of treating the general, two-sided case and also in the sense of obtaining higher order expansions) by what has become to be known as the *Banach algebra method*, essentially the use of Gelfand transforms and the Wiener–Lévy–Gelfand theorem. The use of this method can be traced back to one of the first proofs of the discrete renewal theorem in [**5**] and has since been used in [**3**, **6**, **16–18**] and others; a variant that produces expansions was introduced in [**11**, **13**].

It is one aim of the present paper to show that the Banach algebra method can also be applied to random record distributions. This in itself might not be very surprising, but the final result is in our view remarkable as it differs significantly from those obtained for renewal measures, harmonic renewal measures and compound distributions with exponentially decreasing weights. To explain this, note that in all these cases after a finite number of normalization terms (such as c_1x+c_2 in (1·2) and $\log x+c_1$ in (1·3)) we are down to an order of magnitude that depends directly on the rate of tail decrease of μ. Indeed, it is often possible to show with elementary complex variable arguments that an exponentially decreasing tail of μ implies an exponential rate of convergence in some associated limit theorem for ν. A naive interpolation of (1·2), (1·3), (1·4) and Westcott's result would then lead us to expect for (1·1) a major term c_1/x and a remainder whose order depends on the tail of μ. However, we will see below that random record distributions do not fit into this scheme – the expansion will be in negative powers of x, irrespective of the rate of tail decrease of μ. Also, it turns out that in order to handle the random record situation some new arguments are needed. Some of these are of a rather technical nature (see e.g. Lemma 9 below) and might not yet have found their most natural form; we suspect that the additional log-factor in our error bounds (which is not present in the results for the other convolution series mentioned above) is due to this circumstance. In any case we are confident that these techniques considerably enlarge the range of applicability of the Banach algebra method.

In order to state our main result we require some more notation. We will generally assume that μ is *spread out*, i.e. that $\mu^{\star n}$ has a non-vanishing absolutely continuous component for some $n \in \mathbb{N}$, and our assumptions on the rate of tail decrease are of the form

$$\mu((-\infty,-x]) = O(x^{-\gamma}), \quad \mu((x,\infty)) = O(x^{-\gamma}) \quad \text{as } x \to \infty \tag{1·5}$$

for some $\gamma > 1$. If this holds then the moments

$$m_j(\mu) := \int x^j \, \mu(dx), \quad j = 1,\dots,k = k(\gamma) := \lceil \gamma \rceil - 1$$

exist and we can define the *inverse moments* $r_0(\mu),\dots,r_k(\mu)$ associated with μ inductively by

$$r_0(\mu) := 1, \quad r_j(\mu) := -\sum_{l=1}^{j} \binom{j}{l} m_l(\mu)\, r_{j-l}(\mu) \quad \text{for } j = 1,\dots,k. \tag{1·6}$$

The reason for this name will become clear in the course of the proof of our theorem; note that $r_1(\mu) = -m_1(\mu)$. The following theorem gives an expansion of the tails $\nu_{\mathrm{rr}}((-\infty,-x])$, $\nu_{\mathrm{rr}}((x,\infty))$ of random record distributions in terms of negative powers of x, with the coefficients depending on the inverse moments of μ in a simple manner.

THEOREM 1. *Let μ be a spread out probability distribution with $m_1(\mu) > 0$ and assume that* (1·5) *holds for some $\gamma > 1$. Then, as $x \to \infty$, $\nu_{\mathrm{rr}}((-\infty, -x]) = O(x^{-\gamma} \log x)$ and*

$$\nu_{\mathrm{rr}}((x, \infty)) = -\sum_{j=1}^{\lceil\gamma\rceil-1} \frac{r_j(\mu)}{j}\, x^{-j} + O(x^{-\gamma} \log x). \tag{1·7}$$

As long as some of the inverse moments of μ are not equal to zero, the right tail of the random record distribution associated with μ will decrease at a polynomial rate only, even after the unavoidable term $m_1(\mu)/x$ has been subtracted. In this context the exponential distribution plays a special role (as it does in the renewal case, where the remainder term in the corresponding limit theorem vanishes).

Example 2. If μ has density $f_\mu(x) = \lambda e^{-\lambda x}$, $x \geqslant 0$, for some $\lambda > 0$, then $\mu^{\star n}$ is the gamma distribution with shape parameter n and scale parameter λ so that ν_{rr} has density f given by

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{1}{(n+1)n} \frac{\lambda^n x^{n-1}}{(n-1)!}\, e^{-\lambda x} \\ &= \frac{1}{\lambda x^2}(1 - e^{-\lambda x} - \lambda x e^{-\lambda x}). \end{aligned}$$

This implies $\nu_{\mathrm{rr}}((x, \infty)) = (\lambda x)^{-1} + O(e^{-\lambda x})$. Indeed, a simple calculation shows that we have $m_k(\mu) = k!/\lambda^k$ and therefore $r_1(\mu) = -1/\lambda$, $r_k(\mu) = 0$ for all $k > 1$.

Section 2 consists of the proof of Theorem 1, some comments on possible extensions can be found in Section 3.

2. *Proof of the theorem*

We first recall some standard material from the theory of commutative Banach algebras, which we then apply to a class of algebras suitable for our purposes. For the general theory of these structures we refer the reader to the classic [**9**] or one of the many excellent textbooks on functional analysis such as [**19**].

Let $\mathfrak{B}$ be the σ-field of the Borel subsets of the real line, $\mathbb{M}$ denotes the linear space of all σ-additive functions $\mu: \mathfrak{B} \to \mathbb{C}$. Each such complex-valued finite measure μ can be written in the form

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

with non-negative finite measures $\mu_1, \ldots, \mu_4$; this decomposition is unique if we further require that μ_1 and μ_2 respectively μ_3 and μ_4 are concentrated on disjoint Borel sets. The corresponding total variation measure $|\mu|$ is then given by $|\mu| = \mu_1 + \cdots + \mu_4$ and satisfies

$$|\mu|((a, b]) = \sup\left\{\sum_{k=1}^{n} |\mu((c_{k-1}, c_k])| : n \in \mathbb{N},\ a = c_0 \leqslant c_1 \leqslant \cdots \leqslant c_n = b\right\}$$

for all $a, b \in \mathbb{R}$ with $a < b$. We write $m_k(\mu)$ for the kth moment $\int x^k\, \mu(dx)$ provided that $m_k(\mu_i)$ is finite for $i = 1, \ldots, 4$. In this case the associated Fourier transform

$$\hat{\mu}: \mathbb{R} \to \mathbb{C}, \quad \hat{\mu}(\theta) := \int e^{i\theta x}\, \mu(dx),$$

is k times differentiable and $(d^k\hat{\mu}/d\theta^k)(0) = i^k m_k(\mu)$. We mention in passing that the existence of the kth derivative of $\hat{\mu}$ at 0 does not imply the existence of the kth moment, not even for non-negative measures, but that this formula is convenient for the calculation of moments once their existence has been established.

With the norm $\|\mu\|_{\mathrm{TV}} := |\mu|(\mathbb{R})$ the space $\mathbb{M}$ becomes a Banach space. Moreover, the convolution

$$\mu \star \nu(A) := \int \mu(A-x)\,\nu(dx) \quad \text{for all } A \in \mathfrak{B},$$

defines a 'multiplication', which makes $\mathbb{M}$ a commutative Banach algebra with unit element δ_0, the unit measure concentrated at 0. Convolution and moments interact as follows,

$$m_k(\mu \star \nu) = \sum_{j=0}^{k} \binom{k}{j} m_j(\mu)\, m_{k-j}(\nu), \tag{2·1}$$

provided that the moments exist. Comparing (2·1) and (1·6) we see that the inverse moments of μ are simply the moments of some convolution inverse of μ in $\mathbb{M}$ (the precise interpretation involves localization, which we carry out below). Maximal ideals I in $\mathbb{M}$ are either of the form

$$I = I(\theta_0) := \{\mu \in \mathbb{M}: \hat{\mu}(\theta_0) = 0\}$$

for some $\theta_0 \in \mathbb{R}$, or they contain $\mathbb{M}_a$, the set of all absolutely continuous measures (see e.g. [**9**, section 30]). There is a one-to-one correspondence between maximal ideals I and multiplicative functionals $\psi: \mathbb{M} \to \mathbb{C}$, given by $I = \psi^{-1}(\{0\})$, and the Gelfand transform of μ is the function $\tilde{\mu}$ on the set of maximal ideals defined by $\tilde{\mu}(I) = \psi(\mu)$. From the above statement on the structure of the maximal ideals in $\mathbb{M}$ it then follows that $\tilde{\mu}_1 = \tilde{\mu}_2$ implies $\hat{\mu}_1 = \hat{\mu}_2$. As Fourier transforms characterize measures, so do Gelfand transforms: $\mathbb{M}$ is semisimple. It also follows from the multiplicativity of the ψ-functionals that

$$|\tilde{\mu}(I)| \leqslant \|\mu\|_{\mathrm{TV}} \quad \text{for all } \mu \in \mathbb{M} \tag{2·2}$$

for all maximal ideals I.

We now fix a real number $\gamma > 1$. Let $\mathbb{M}_{\mathrm{r}}(\gamma)$ be the set of all $\mu \in \mathbb{M}$ with the property $|\mu|((x, x+1]) = O(x^{-\gamma})$ as $x \to \infty$. With

$$\|\mu\|_{r,\gamma} := \|\mu\|_{\mathrm{TV}} + 2^{\gamma} \sup_{x \geqslant 0} (1+x)^{\gamma}\, |\mu|((x, x+1])$$

this is a Banach algebra and to any maximal ideal I in this space there exists a maximal ideal I_0 in $\mathbb{M}$ such that $I = I_0 \cap \mathbb{M}_{\mathrm{r}}(\gamma)$ (see [**17**]). Reflection at 0 defines an operator $S: \mathbb{M} \to \mathbb{M}$, $S(\mu)(A) := \mu(-A)$ and the corresponding spaces of measures characterized by the behaviour of $|\mu|([x, x+1))$ as $x \to -\infty$ are $\mathbb{M}_l(\gamma) := \{\mu \in \mathbb{M}: S(\mu) \in \mathbb{M}_{\mathrm{r}}(\gamma)\}$. The space of main interest to us is $\mathbb{M}(\gamma) := \mathbb{M}_l(\gamma) \cap \mathbb{M}_{\mathrm{r}}(\gamma)$, with norm $\|\mu\|_{\gamma} := \|S(\mu)\|_{r,\gamma} + \|\mu\|_{r,\gamma}$. This is again a Banach algebra and a little separate argument shows that the maximal ideals of $\mathbb{M}(\gamma)$ again arise as the intersections of maximal ideals in $\mathbb{M}$ with $\mathbb{M}(\gamma)$. In particular, the range of the Gelfand transform of some $\mu \in \mathbb{M}(\gamma)$ with respect to $\mathbb{M}(\gamma)$ is a subset of the range of the Gelfand transform of μ regarded as an element of $\mathbb{M}$. This range is important in view of the Wiener–Lévy–Gelfand theorem, which we need in the following form.

Proposition 3. *Let $\mu \in \mathbb{M}(\gamma)$ and $U \subset \mathbb{C}$, U open, be such that the range of the Gelfand transform of μ is contained in U. Let $\Psi: U \to \mathbb{C}$ be an analytic function. Then there exists a unique element ν of $\mathbb{M}(\gamma)$ with the property $\hat{\nu} = \Psi \circ \hat{\mu}$.*

Of course, the same statement holds if $\mathbb{M}(\gamma)$ is replaced by $\mathbb{M}$ throughout provided the Gelfand transform refers to this space too. In view of the above inclusion relation and the connection to Fourier transforms we will generally consider the range of the Gelfand transform with respect to $\mathbb{M}$. We will use Proposition 3 with the following (U, Ψ)-pairs,

$$U_1 := \{z \in \mathbb{C}: z \neq 0\}, \quad \Psi_1(z) := \frac{1}{z},$$
$$U_2 := \{z \in \mathbb{C}: \mathrm{Re}(z) > 0 \quad \text{or} \quad \mathrm{Im}\,(z) \neq 0\}, \quad \Psi_2(z) := \log(z),$$
$$U_3 := \{z \in \mathbb{C}: |z| < 1\}, \quad \Psi_3(z) := \sum_{k=1}^{\infty} \frac{1}{k(k+1)}\, z^k.$$

With the logarithm we always mean the canonical version, which is real for positive real arguments and analytic on U_2. The function Ψ_3 is obviously the one that relates directly to random record measures. Note that we can extend Ψ_3 continuously to the closure of U_3 by

$$\Psi_3(z) = 1 + \left(\frac{1}{z} - 1\right) \log(1-z) \quad \text{if } |z| \leqslant 1,\ z \notin \{0, 1\},\ \Psi_3(0) = 0,\ \Psi_3(1) = 1,$$

and that, with this extension,

$$\nu_{\mathrm{rr}}(\mu)\hat{\ }(\theta) = \Psi_3(\hat{\mu}(\theta)) \quad \text{for all } \theta \in \mathbb{R} \tag{2·3}$$

for all probability measures μ on the real line. In particular, if μ_0 denotes the exponential distribution with mean 1 then

$$\nu_{\mathrm{rr}}(\mu_0)\hat{\ }(\theta) = 1 + i\theta \log\left(1 - \frac{1}{i\theta}\right) \quad \text{for all } \theta \neq 0.$$

In order to cope with exceptional points in the range of $\tilde{\mu}$ we need to be able to localize. The existence of suitable partitions of unity is guaranteed by the following auxiliary result.

Lemma 4. *For any $\alpha > 0$ there exists an element*

$$\rho_\alpha \in \mathbb{M}_a(\infty) := \mathbb{M}_a \cap \bigcap_{k=1}^{\infty} \mathbb{M}(k)$$

with $m_0(\rho_\alpha) = 1$, $m_k(\rho_\alpha) = 0$ for all $k \in \mathbb{N}$ and

$$\hat{\rho}_\alpha(\mathbb{R}) \subset [0,1], \quad \hat{\rho}_\alpha(\theta) = \begin{cases} 1, & \text{if } |\theta| \leqslant \alpha, \\ 0, & \text{if } |\theta| \geqslant 2\alpha. \end{cases}$$

Moreover, if $\mu \in \mathbb{M}$ is such that $\mu((x,\infty)) = O(x^{-\gamma}\log x)$ for some $\gamma > 1$ as $x \to \infty$ then also $\rho_\alpha \star \mu((x,\infty)) = O(x^{-\gamma}\log x)$ as $x \to \infty$ and the same statement holds with (x,∞) replaced by $(-\infty,-x]$.

Proof. It is well-known that there exists an infinitely often differentiable function $\phi: \mathbb{R} \to \mathbb{C}$ with the properties required for $\hat{\rho}_\alpha$. The inversion formula for characteristic functions implies that there is a $\rho_\alpha \in \mathbb{M}_a$ with Fourier transform ϕ and that

$\int |x|^k \, |\rho_\alpha|(dx) < \infty$ for all $k \in \mathbb{N}$. From this $|\rho_\alpha|\big((-\infty,-x] \cup (x,\infty)\big) = o(x^{-k})$ as $x \to \infty$ follows with Markov's inequality, i.e. $\rho_\alpha \in \mathbb{M}(k)$. The same argument can be applied to $\theta \to \hat{\rho}_\alpha(\theta)(i\theta)^l$ for any $l \in \mathbb{N}$, which therefore must be the transform of an element of $\mathbb{M}_a(\infty)$ too. Finally, we obviously have $\big(d^k\hat{\rho}_\alpha/d\theta^k\big)(0) = 0$ and hence $m_k(\rho_\alpha) = 0$ for all $k \in \mathbb{N}$.

For the proof of the last statement of the lemma it is obviously enough to consider the right tail. We split the range of integration,

$$\rho_\alpha \star \mu((x,\infty)) = \int_{(-\infty,x/2]} \rho_\alpha((x-y,\infty))\,\mu(dy) + \int_{(x/2,\infty)} \rho_\alpha((x-y,\infty))\,\mu(dy).$$

For the first term we use

$$\Big|\int_{(-\infty,x/2]} \rho_\alpha((x-y,\infty))\,\mu(dy)\Big| \leqslant \big|\rho_\alpha\big|((x/2,\infty))\ \|\mu\|_{\mathrm{TV}},$$

which is $O(x^{-\eta})$ for all $\eta > 0$. For the second term we use Fubini's theorem (integration by parts) to obtain

$$\begin{aligned}&\int_{(x/2,\infty)} \rho_\alpha((x-y,\infty))\,\mu(dy)\\ &\qquad\qquad = \rho_\alpha((x/2,\infty))\,\mu((x/2,\infty)) + \int_{(-\infty,x/2)} \mu((x-z,\infty))\,\rho_\alpha(dz)\end{aligned}$$

and then bound the individual terms,

$$\begin{aligned}|\rho_\alpha((x/2,\infty))\,\mu((x/2,\infty))| &\leqslant \|\rho_\alpha\|_{\mathrm{TV}}\,|\mu((x/2,\infty))|,\\ \Big|\int_{(-\infty,x/2)} \mu((x-z,\infty))\,\rho_\alpha(dz)\Big| &\leqslant \|\rho_\alpha\|_{\mathrm{TV}} \sup_{y\geqslant x/2} |\mu((x/2,\infty))|,\end{aligned}$$

which again results in the upper bound $O(x^{-\gamma}\log x)$.

The *tail function* F_μ associated with some $\mu \in \mathbb{M}$ is given by

$$F_\mu : \mathbb{R} \to \mathbb{C}, \qquad F_\mu(x) := \begin{cases} \mu((x,\infty)), & \text{if } x \geqslant 0,\\ -\mu((-\infty,x]), & \text{if } x < 0.\end{cases}$$

If this function is integrable then we can define a new measure $\Sigma\mu \in \mathbb{M}_a$, the *tail measure* associated with μ, by

$$\Sigma\mu(A) = \int_A F_\mu(x)\,dx \quad \text{for all } A \in \mathfrak{B},$$

and we can regard Σ as a linear operator on

$$\mathrm{Dom}\,(\Sigma) := \Big\{\mu \in \mathbb{M}\colon \int |F_\mu(x)|\,dx < \infty\Big\}$$

with values in $\mathbb{M}$. It is easy to check that μ is in $\mathrm{Dom}\,(\Sigma)$ if μ is a probability measure with finite first moment, or generally if $\mu \in \mathbb{M}(\gamma)$ for some $\gamma > 2$.

The proofs of the following statements are either elementary or contained in [**12**, **13**] and are therefore omitted. Remember that $m_k(\mu)$ denotes the kth moment $\int x^k\,\mu(dx)$ of μ. We write Σ^k for the k-fold iteration of Σ, where Σ^0 is understood to be the identity.

LEMMA 5.

(i) *If* $\mu \in \mathbb{M}(\gamma+1)$ *then* $\Sigma\mu \in \mathbb{M}(\gamma)$.

(ii) *For any* $\mu \in \mathrm{Dom}\,(\Sigma)$,

$$(\Sigma\mu)\hat{}\,(\theta) = \frac{\hat{\mu}(\theta) - \hat{\mu}(0)}{i\theta} \quad \text{if } \theta \neq 0, \quad (\Sigma\mu)\hat{}\,(0) = m_1(\mu).$$

(iii) *Suppose that* $\mu_1, \mu_2 \in \mathrm{Dom}\,(\Sigma)$ *and* $\mu_1(\mathbb{R}) = 0$. *Then* $\Sigma(\mu_1 \star \mu_2) = (\Sigma\mu_1) \star \mu_2$.

(iv) *If* $\mu \in \mathbb{M}(\gamma)$ *is such that* $\Sigma\mu \in \mathbb{M}(\gamma)$, *then* $\mu((-\infty, -x]) = O(x^{-\gamma})$ *and* $\mu((x,\infty)) = O(x^{-\gamma})$ *as* $x \to \infty$.

(v) *If* $\mu \in \mathbb{M}(\gamma)$ *then, for* $j \in \mathbb{N}_0, k \in \mathbb{N}$ *with* $j + k < \gamma$,

$$m_{j-1}(\Sigma^k \mu) = \frac{1}{j}\, m_j(\Sigma^{k-1}\mu).$$

In particular, for $k < \gamma - 1$,

$$(\Sigma^k \mu)(\mathbb{R}) = \frac{1}{k!}\, m_k(\mu).$$

We next introduce a reference family of measures with known tail decrease. For any $j \in \mathbb{N}$ let $f_j\colon \mathbb{R} \to \mathbb{R}$ be defined by $f_j(x) := 0$ for $x < 0$ and

$$f_j(x) := j!\, e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{(k+j+1)!} = \frac{j!}{x^{j+1}} \Big(1 - e^{-x} \sum_{k=0}^{j} \frac{x^k}{k!}\Big)$$

for $x > 0$, for definiteness we take $f_j(0)$ to be the right hand limit $1/(j+1)$. Let ν_j be the non-negative measure with Lebesgue density f_j, let μ_0 be the exponential distribution with mean 1. From Example 2 we know that ν_1 is the random record distribution associated with μ_0; for general j we have

$$\nu_j = \sum_{k=1}^{\infty} \frac{j!}{k(k+1)\cdots(k+j)}\, \mu_0^{\star k}.$$

LEMMA 6. *Let* $j \in \mathbb{N}$. *Then* ν_j *is a finite measure with total mass* $1/j$ *and Fourier transform*

$$\hat{\nu}_j(\theta) = \sum_{k=0}^{j-1} \frac{1}{j-k} (i\theta)^k + (i\theta)^j \log\Big(1 - \frac{1}{i\theta}\Big).$$

Moreover, $\nu_j \in \mathbb{M}(j+1)$ *and* $\Sigma^l \nu_j = \nu_{j-l}$ *for* $l = 1, \ldots, j-1$.

Proof. We have

$$\begin{aligned} f_j'(x) &= -j!\, e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{(k+j+1)!} + j!\, e^{-x} \sum_{k=1}^{\infty} \frac{k x^{k-1}}{(k+j+1)!} \\ &= -j!\, e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{(k+j+2)!} \big(k + j + 2 - (k+1)\big) \\ &= -f_{j+1}(x) \quad \text{for all } x > 0, \end{aligned}$$

which implies $\Sigma\nu_{j+1} = \nu_j$. This together with the known formula for $\hat{\nu}_1$ can be used to prove the formula for $\hat{\nu}_j$ by induction, and this in turn delivers the total mass.

The next lemma shows that the localizing functions introduced above do not change the tail behaviour of the reference measures on the scale that is of interest to us.

Lemma 7. *With ρ_α and ν_j as above we have, for all $\eta > 0$ and as $x \to \infty$,*

$$\rho_\alpha \star \nu_j((-\infty, -x]) = o(x^{-\eta}), \quad \rho_\alpha \star \nu_j((x,\infty)) - \nu_j((x,\infty)) = o(x^{-\eta}).$$

Proof. Let $k \in \mathbb{N}$ be such that $j + k > \eta$. We apply Lemma 5(iii) k times and use $\Sigma^l(\rho_\alpha - \delta_0)(\mathbb{R}) = 0$ for $l = 0, \dots, k$, $\nu_j = \Sigma^k \nu_{j+k}$ to obtain

$$(\rho_\alpha - \delta_0) \star \nu_j = \Sigma^k \rho_\alpha \star \nu_{j+k} - \sum_{l=1}^{k} \frac{1}{j+l}\, \Sigma^l \rho_\alpha.$$

Both factors of the first term on the right-hand side are in $\mathbb{M}(j+k+1)$ and so are the terms in the sum by Lemma 5(i) and Lemma 4, hence $(\rho_\alpha - \delta_0) \star \nu_j \in \mathbb{M}(j+k+1)$. This implies that the associated tails decrease at least as fast as x^{-j-k}, which is $o(x^{-\eta})$.

We will use operational calculus in $\mathbb{M}(\gamma)$ mainly to get rid of certain error terms in various preliminary steps of the proof of our theorem. For the expansion itself the rearrangement given in Lemma 8 below together with the bound in Lemma 9 will be crucial. The proof of the latter is somewhat lengthy, but uses elementary arguments only; it is here that the log-factors in the error bounds of Theorem 1 appear.

Lemma 8. *Let $\mu \in \mathbb{M}(\gamma)$ for some $\gamma > 1$ and put $k := \lceil \gamma \rceil - 2$. Then $\Sigma^j \mu$ is a finite measure for $j = 0, \dots, k$. Let $s_j := \Sigma^j \mu(\mathbb{R})$ and let ν_j be as in Lemma 6. Then, for all $\theta \in \mathbb{R}$,*

$$\begin{aligned}(\hat{\mu}(\theta) - s_0)\, i\theta \log\Big(1 - \frac{1}{i\theta}\Big) &= ((\Sigma^k \mu)\hat{\ }(\theta) - s_k)\Big(\hat{\nu}_{k+1}(\theta) - \frac{1}{k+1}\Big) \\ &\quad - \sum_{j=0}^{k-1} \frac{1}{j+1} ((\Sigma^j \mu)\hat{\ }(\theta) - s_j) + \sum_{j=1}^{k} s_j \Big(\hat{\nu}_{j+1}(\theta) - \frac{1}{j+1}\Big).\end{aligned}$$

Proof. For $k = 0$ this is immediate from the formula for the transform of ν_1. The induction step will follow if we can show that

$$\begin{aligned}&((\Sigma^k \mu)\hat{\ }(\theta) - s_k)\Big(\hat{\nu}_{k+1}(\theta) - \frac{1}{k+1}\Big) + s_k\Big(\hat{\nu}_{k+1}(\theta) - \frac{1}{k+1}\Big) \\ &\qquad = ((\Sigma^{k-1}\mu)\hat{\ }(\theta) - s_{k-1})\Big(\hat{\nu}_k(\theta) - \frac{1}{k}\Big) + \frac{1}{k}((\Sigma^{k-1}\mu)\hat{\ }(\theta) - s_{k-1}).\end{aligned}$$

From Lemma 5(iii) we obtain for arbitrary $\mu_1, \mu_2 \in \mathrm{Dom}\,(\Sigma)$

$$\begin{aligned}&((\Sigma\mu_1)\hat{\ }(\theta) - (\Sigma\mu_1)\hat{\ }(0))(\hat{\mu}_2(\theta) - \hat{\mu}_2(0)) + (\Sigma\mu_1)\hat{\ }(0)(\hat{\mu}_2(\theta) - \hat{\mu}_2(0)) \\ &\qquad = (\hat{\mu}_1(\theta) - \hat{\mu}_1(0))((\Sigma\mu_2)\hat{\ }(\theta) - (\Sigma\mu_2)\hat{\ }(0)) + (\Sigma\mu_2)\hat{\ }(0)(\hat{\mu}_1(\theta) - \hat{\mu}_1(0)),\end{aligned}$$

so the equation follows on using $\Sigma\nu_{k+1} = \nu_k$ and $\hat{\nu}_k(0) = 1/k$.

Lemma 9. *Suppose that $\gamma > 1$ and let $k := \lceil \gamma \rceil - 2$. Let μ be a measure with bounded density f satisfying*

$$f(x) = O(|x|^{-\gamma}) \quad \text{for } x \to \pm\infty$$

and let s_j be the total mass of $\Sigma^j \mu$, $j = 0, \dots, k$. Then, as $x \to \infty$ and with ν_{k+1} as in

Lemma 6,

$$\left((\Sigma^k\mu - s_k\delta_0) \star (\nu_{k+1} - \frac{1}{k+1}\,\delta_0) - \sum_{j=0}^{k-1} \frac{1}{j+1}\Sigma^j\mu \right) (I(x)) = O(x^{-\gamma}\log x)$$

for $I(x) = (x,\infty)$ *and for* $I(x) = (-\infty, -x]$.

Proof. We first consider the case $I(x) = (x,\infty)$. Let $x > 0$ and put

$$\begin{aligned} A(x) &:= \{(y,z) \in \mathbb{R}^2 : 0 \leqslant y \leqslant x,\ 0 \leqslant z \leqslant x,\ y+z > x\},\\ B(x) &:= \{(y,z) \in \mathbb{R}^2 : y < 0,\ x < z < x - y\}. \end{aligned}$$

Further, $\mu \otimes \nu$ denotes the product of any two measures μ and ν in $\mathbb{M}$. Using the fact that ν_{k+1} is concentrated on $(0,\infty)$ we obtain

$$\begin{aligned} &(\Sigma^k\mu - s_k\delta_0) \star (\nu_{k+1} - \frac{1}{k+1}\delta_0)((x,\infty))\\ &\quad = \Sigma^k\mu \star \nu_{k+1}((x,\infty)) - s_k\,\nu_{k+1}((x,\infty)) - \frac{1}{k+1}\Sigma^k\mu((x,\infty))\\ &\quad = \Sigma^k\mu \otimes \nu_{k+1}(\{(y,z) \in \mathbb{R}^2 : y+z > x\})\\ &\qquad -\Sigma^k\mu \otimes \nu_{k+1}(\{(y,z) \in \mathbb{R}^2 : z > x\})\\ &\qquad -\Sigma^k\mu \otimes \nu_{k+1}(\{(y,z) \in \mathbb{R}^2 : y > x\})\\ &\quad = \Sigma^k\mu \otimes \nu_{k+1}(A(x)) - \Sigma^k\mu \otimes \nu_{k+1}(B(x))\\ &\qquad -\Sigma^k\mu \otimes \nu_{k+1}((x,\infty)\times(x,\infty)). \end{aligned}$$

For the last term the desired rate follows immediately from

$$\Sigma^k\mu((x,\infty)) = O(x^{k+1-\gamma}), \quad \nu_{k+1}((x,\infty)) = O(x^{-k-1}).$$

For the purposes of obtaining upper bounds we may replace the densities of $\Sigma^k\mu$ and ν_{k+1} by suitable multiples of $x \to (1+|x|)^{-\gamma+k}$ and, for $x > 0$, $x \to (1+x)^{-k-2}$ respectively. In particular, for some suitable constant c and with the elementary inequality

$$(1+x+y)^{k+1} - (1+x)^{k+1} \leqslant y\,(k+1)\,(1+x+y)^k \quad \text{for all } x, y \geqslant 0$$

we obtain

$$\begin{aligned} |\Sigma^k\mu \otimes \nu_{k+1}(B(x))| &\leqslant c \int_{-\infty}^{0}\int_{x}^{x-y} \frac{1}{(1+z)^{k+2}}\,dz\,\frac{1}{(1+|y|)^{\gamma-k}}\,dy\\ &= \frac{c}{(k+1)(1+x)^{k+1}} \int_0^\infty \frac{(1+x+y)^{k+1} - (1+x)^{k+1}}{(1+x+y)^{k+1}(1+y)^{\gamma-k}}\,dy\\ &\leqslant \frac{c}{(1+x)^{k+1}} \int_0^\infty \frac{y}{(1+x+y)(1+y)^{\gamma-k}}\,dy. \end{aligned}$$

We need the upper bound $O(x^{k+1-\gamma}\log x)$ for the integral. For the range from 0 to x we use

$$\int_0^x \frac{y}{(1+x+y)(1+y)^{\gamma-k}}\,dy \leqslant \frac{1}{1+x}\int_0^x (1+y)^{k+1-\gamma}\,dy$$

and consider the cases $k+1 < \gamma < k+2$ and $\gamma = k+2$ separately: the first leads to

$$\int_0^x (1+y)^{k+1-\gamma}\,dy = O(x^{k+2-\gamma}),$$

in the second case we get

$$\int_0^x (1+y)^{k+1-\gamma}\,dy = O(\log x),$$

i.e. in both cases the required overall rate $O(x^{k+1-\gamma}\log x)$ results. For the remaining range from x to ∞ the required rate is immediate from

$$\begin{aligned}\int_x^\infty \frac{y}{(1+x+y)(1+y)^{\gamma-k}}\,dy &\leqslant \int_x^\infty (1+y)^{k-\gamma}\,dy \\ &= \frac{1}{\gamma-k-1}\,(1+x)^{k+1-\gamma},\end{aligned}$$

where we used $\gamma - k > 1$.

For $A(x)$ we use $A(x) = A_1(x) + A_2(x)$ with

$$A_1(x) := \bigl\{(y,z)\in A(x)\colon y \leqslant x/2\bigr\},\quad A_2(x) := \bigl\{(y,z)\in A(x)\colon y > x/2\bigr\}.$$

Bounding the densities as for $B(x)$ for the first of these we obtain

$$\begin{aligned}|\Sigma^k\mu\otimes\nu_{k+1}(A_1(x))| &\leqslant c\int_0^{x/2}\int_{x-y}^x \frac{1}{(1+z)^{k+2}}\,dz\,\frac{1}{(1+y)^{\gamma-k}}\,dy \\ &\leqslant \frac{2^{k+2}\,c}{(1+x)^{k+2}}\int_0^{x/2}\frac{y}{(1+y)^{\gamma-k}}\,dy.\end{aligned}$$

As in the above analysis of $B(x)$ we obtain the rate $O(\log x)$ for the integral if $\gamma = k+2$ and $O(x^{k+2-\gamma})$ if $k+1<\gamma<k+2$, so this term has the desired rate $O(x^{-\gamma}\log x)$ too.

For the remaining part $A_2(x)$ we first consider the case $k=0$. As for $A_1(x)$,

$$\begin{aligned}|\mu\otimes\nu_1(A_2(x))| &\leqslant c\int_{x/2}^x\int_{x-y}^\infty \frac{1}{(1+z)^2}\,dz\,\frac{1}{(1+y)^\gamma}\,dy \\ &= c\int_{x/2}^x \frac{1}{1+x-y}\,\frac{1}{(1+y)^\gamma}\,dy \\ &\leqslant \frac{2^\gamma\,c}{(1+x)^\gamma}\int_0^{x/2}\frac{1}{1+y}\,dy,\end{aligned}$$

which is $O(x^{-\gamma}\log x)$.

For $k>0$ a suitable integration by parts will be crucial. In extension of the definition preceding Lemma 6 let $f_0\colon[0,\infty)\to\mathbb{R}$ be defined by $f_0(0) := 1$ and

$$f_0(x) := \frac{1}{x}\,(1-e^{-x})\quad \text{for all } x>0.$$

(In contrast to the functions f_k with $k>0$ this is not the density of a finite measure.) Remember that f_j is the density of ν_j for $j=1,2,\ldots$, and note that $f_j' = -f_{j+1}$ also holds for $j=0$ where the derivative refers to the right derivative at $x=0$. Also,

$f_j(0) = 1/(j+1)$ holds for all non-negative integers j. Then

$$\begin{aligned}
\Sigma^k\mu \otimes \nu_{k+1}(A_2(x)) &= \int_{x/2}^x \int_{x-y}^x f_{k+1}(z)\,dz\; \Sigma^{k-1}\mu((y,\infty))\,dy \\
&= \int_{x/2}^x \bigl(f_k(x-y) - f_k(x)\bigr)\,\Sigma^{k-1}\mu((y,\infty))\,dy \\
&= \sum_{j=1}^k \left(f_{k-j}(x-y) - \frac{1}{j!}\,y^j f_k(x)\right)\Sigma^{k-j}\mu((y,\infty))\Big|_{y=x/2}^x \\
&\quad + \int_{x/2}^x \left(f_0(x-y) - \frac{1}{k!}\,y^k f_k(x)\right)\mu(dy) \\
&= \sum_{j=1}^k \frac{1}{k+1-j}\Sigma^{k-j}\mu((x,\infty)) - \sum_{j=1}^k f_{k-j}\Bigl(\frac{x}{2}\Bigr)\Sigma^{k-j}\mu\Bigl(\Bigl(\frac{x}{2},\infty\Bigr)\Bigr) \\
&\quad - \sum_{j=1}^k \frac{1}{j!}\,x^j f_k(x)\Sigma^{k-j}\mu((x,\infty)) + \sum_{j=1}^k \frac{1}{j!}\,\Bigl(\frac{x}{2}\Bigr)^j f_k(x)\Sigma^{k-j}\mu\Bigl(\Bigl(\frac{x}{2},\infty\Bigr)\Bigr) \\
&\quad + \int_{x/2}^x \left(f_0(x-y) - \frac{1}{k!}\,y^k f_k(x)\right)\mu(dy).
\end{aligned}$$

We consider these terms individually and start at the end: as for $k = 0$,

$$\begin{aligned}
\Bigl|\int_{x/2}^x f_0(x-y)\,\mu(dy)\Bigr| &\leqslant \frac{c}{(1+x)^\gamma}\int_0^{x/2}\frac{1}{1+y}\,dy, \\
\Bigl|\int_{x/2}^x y^k\, f_k(x)\,\mu(dy)\Bigr| &\leqslant \frac{c}{(1+x)^{k+1}}\int_{x/2}^\infty (1+y)^{k-\gamma}\,dy,
\end{aligned}$$

which together yields the rate $O(x^{-\gamma}\log x)$ for the integral. For the terms appearing in the sums, with the exception of the first sum, this same rate follows from

$$f_j(x) = O(x^{-j-1}),\ \Sigma^j\mu((x,\infty)) = O(x^{-\gamma+j+1}) \quad \text{for } j = 0,\dots,k.$$

In total a cancellation occurs and we arrive at

$$\Sigma^k\mu \otimes \nu_{k+1}(A_2(x)) - \sum_{j=0}^{k-1}\frac{1}{j+1}\Sigma^j\mu((x,\infty)) = O(x^{-\gamma}\log x).$$

This means that the statement of the lemma has been proved for $I(x) = (x,\infty)$.

For the left tail we use essentially the same arguments. With

$$C(x) \coloneqq \{(y,z)\in\mathbb{R}^2 : y \leqslant -x,\ z \geqslant -x-y\}$$

and using the fact that ν_{k+1} is concentrated on $(0,\infty)$ we obtain for $x > 0$

$$\begin{aligned}
&(\Sigma^k\mu - s_k\delta_0)\star\left(\nu_{k+1} - \frac{1}{k+1}\delta_0\right)((-\infty,-x]) \\
&\quad = \Sigma^k\mu\star\nu_{k+1}((-\infty,-x]) - \frac{1}{k+1}\,\Sigma^k\mu((-\infty,-x]) \\
&\quad = \Sigma^k\mu\otimes\nu_{k+1}(\{(y,z)\in\mathbb{R}^2 : y+z \leqslant -x\}) \\
&\qquad -\Sigma^k\mu\otimes\nu_{k+1}(\{(y,z)\in\mathbb{R}^2 : y \leqslant -x\}) \\
&\quad = -\Sigma^k\mu\otimes\nu_{k+1}(C(x)).
\end{aligned}$$

We split $C(x) = C_1(x) + C_2(x)$ with

$$C_1(x) := \big\{(y,z) \in C(x)\colon y < -2x\big\}, \quad C_2(x) := \big\{(y,z) \in C(x)\colon y \geqslant -2x\big\}.$$

For the $C_1(x)$-part we use, similar to the argument for $A_1(x)$ in the right tail situation,

$$\begin{aligned}
|\Sigma^k\mu \otimes \nu_{k+1}(C_1(x))| &\leqslant c \int_{-\infty}^{-2x}\int_{-x-y}^{\infty} \frac{1}{(1+z)^{k+2}}\,dz\,\frac{1}{(1+|y|)^{\gamma-k}}\,dy\\
&= c\int_{2x}^{\infty} \frac{1}{(k+1)(1+y-x)^{k+1}}\,\frac{1}{(1+y)^{\gamma-k}}\,dy\\
&\leqslant c\int_{x}^{\infty}\frac{1}{(k+1)(1+y)^{\gamma+1}}\,dy,
\end{aligned}$$

which is $O(x^{-\gamma})$. For $k = 0$ we further have

$$\begin{aligned}
|\mu \otimes \nu_1(C_2(x))| &\leqslant c \int_{-2x}^{-x}\int_{-x-y}^{\infty} \frac{1}{(1+z)^{2}}\,dz\,\frac{1}{(1+|y|)^{\gamma}}\,dy\\
&= c\int_{x}^{2x} \frac{1}{1+y-x}\,\frac{1}{(1+y)^{\gamma}}\,dy\\
&\leqslant \frac{c}{(1+x)^{\gamma}}\int_{0}^{x}\frac{1}{1+y}\,dy,
\end{aligned}$$

which is $O(x^{-\gamma}\log x)$. If $k > 0$ we again use a k-fold integration by parts,

$$\begin{aligned}
\Sigma^k\mu \otimes \nu_{k+1}(C_2(x)) &= -\int_{-2x}^{-x}\int_{-x-y}^{\infty} f_{k+1}(z)\,dz\,\Sigma^{k-1}\mu((-\infty,y]))\,dy\\
&= -\int_{-2x}^{-x} f_k(-y-x)\,\Sigma^{k-1}\mu((-\infty,y])\,dy\\
&= -\sum_{j=1}^{k} f_{k-j}(-y-x)\,\Sigma^{k-j}\mu((-\infty,y])\Big|_{y=-2x}^{-x} + \int_{-2x}^{-x} f_0(-y-x)\,\mu(dy)\\
&= -\sum_{j=1}^{k}\frac{1}{k+1-j}\Sigma^{k-j}\mu((-\infty,-x]) + \sum_{j=1}^{k} f_{k-j}(x)\,\Sigma^{k-j}\mu((-\infty,-2x])\\
&\quad + \int_{-2x}^{-x} f_0(-y-x)\,\mu(dy).
\end{aligned}$$

For the integral we use

$$\Big|\int_{-2x}^{-x} f_0(-y-x)\,\mu(dy)\Big| \leqslant c\int_{x}^{2x}\frac{1}{1+y-x}\frac{1}{(1+y)^{\gamma}}\,dy$$

and the rate $O(x^{-\gamma}\log x)$ follows with the now familiar arguments. The second sum can be handled as the $A_2(x)$-case. Again, overall a cancellation occurs and we arrive at

$$-\Sigma^k\mu \otimes \nu_{k+1}\big(C_2(x)\big) - \sum_{j=0}^{k-1}\frac{1}{j+1}\Sigma^j\mu((-\infty,-x]) = O(x^{-\gamma}\log x),$$

which completes the proof of the second statement.

We now relate the above constructions to the random record situation. Throughout the following let μ be a probability distribution that satisfies the conditions of Theorem 1. Assumption (1·5) means that $\Sigma\mu \in \mathbb{M}(\gamma)$ which in turn gives $\mu \in \mathbb{M}(\gamma)$ as μ is non-negative. Further, we may assume that $m_1(\mu) = 1$, which leads to some simplification in the notation below. To see this it is enough to check that random record distributions and the assertion of Theorem 1 behave in an equivariant manner under transformations $\mathbb{R} \to \mathbb{R}$, $x \mapsto c \cdot x$ with $c > 0$.

We claim that for any maximal ideal I in $\mathbb{M}$ we have

$$I \supset \mathbb{M}_a \quad \Rightarrow \quad |\tilde{\mu}(I)| < 1. \tag{2·4}$$

For the proof of (2·4) let $l \in \mathbb{N}$ be such that $\mu^{\star l}$ has a non-vanishing absolutely continuous part. The singular part then satisfies $\left\|(\mu^{\star l})_{\text{sing}}\right\|_{\text{TV}} < 1$, so (2·4) follows on using (2·2) and

$$|\tilde{\mu}(I)|^l = |(\mu^{\star l})\tilde{}(I)| = |((\mu^{\star l})_{\text{sing}})\tilde{}(I)| \leqslant \|(\mu^{\star l})_{\text{sing}}\|_{\text{TV}} \; < \; 1.$$

Let μ_0 be the exponential distribution with parameter 1 and let $\nu_{\text{rr}}(\mu)$ and $\nu_{\text{rr}}(\mu_0)$ be the random record distributions associated with μ and μ_0 respectively. From (2·3) we obtain the following formula for the Fourier transform of the difference of these two distributions:

$$\begin{aligned}(\nu_{\text{rr}}(\mu) - \nu_{\text{rr}}(\mu_0))\hat{}(\theta) &= \boldsymbol{\Psi}_3(\hat{\mu}(\theta)) - \boldsymbol{\Psi}_3(\hat{\mu}_0(\theta)) \\ &= \left(\frac{1}{\hat{\mu}(\theta)} - 1\right) \log(1 - \hat{\mu}(\theta)) - i\theta \log\left(1 - \frac{1}{i\theta}\right). \end{aligned} \tag{2·5}$$

Here the last expression has to be interpreted by continuous extension for $\theta = 0$ or those θ-values with $\hat{\mu}(\theta) = 0$ or $\hat{\mu}(\theta) = 1$ (see the discussion of $\boldsymbol{\Psi}_3$ above). Now let $\alpha > 0$ be small enough for

$$\inf_{|\theta| \leqslant 8\alpha} \operatorname{Re}(\hat{\mu}(\theta)) > 0 \tag{2·6}$$

to hold; this is possible because of $\hat{\mu}(0) = 1$ and the continuity of $\hat{\mu}$ at 0. As μ is non-lattice we then also have

$$\sup_{|\theta| \geqslant \alpha} \left|\hat{\mu}(\theta)\right| < 1. \tag{2·7}$$

We now decompose the difference of the record measures given in (2·5) on using the localizing functions introduced in Lemma 4:

$$(\nu_{\text{rr}}(\mu) - \nu_{\text{rr}}(\mu_0))\hat{}(\theta) = \phi_1(\theta) - \phi_2(\theta) + \phi_3(\theta) \tag{2·8}$$

with

$$\begin{aligned}
\phi_1(\theta) &\coloneqq (1 - \hat{\rho}_{2\alpha}(\theta))\boldsymbol{\Psi}_3((1 - \hat{\rho}_\alpha(\theta))\hat{\mu}(\theta)), \\
\phi_2(\theta) &\coloneqq (1 - \hat{\rho}_{2\alpha}(\theta))\boldsymbol{\Psi}_3((1 - \hat{\rho}_\alpha(\theta))\hat{\mu}_0(\theta)), \\
\phi_3(\theta) &\coloneqq \hat{\rho}_{2\alpha}(\theta)\left(\left(\frac{1}{\hat{\mu}(\theta)} - 1\right) \log\left(1 - \hat{\mu}(\theta)\right) - i\theta \log\left(1 - \frac{1}{i\theta}\right)\right).
\end{aligned}$$

By construction,

$$|(1 - \hat{\rho}_\alpha(\theta))\hat{\mu}(\theta)| < 1 \quad \text{for all } \theta \in \mathbb{R},$$

which means that the range of the Fourier transform of $(\delta_0 - \rho_\alpha) \star \mu$ is contained in

U_3. If I is a maximal ideal containing $\mathbb{M}_a$ then (2·4) together with $\rho_\alpha \in \mathbb{M}_a$ implies

$$|((\delta_0 - \rho_\alpha) \star \mu)\tilde{}(I)| = |\tilde{\mu}(I)| < 1.$$

Hence the range of the Gelfand transform of the measure $(\delta_0 - \rho_\alpha) \star \mu$ is contained in U_3, so Proposition 3 implies the existence of an element η_1 of $\mathbb{M}(\gamma)$ with Fourier transform $\hat{\eta}_1(\theta) = \boldsymbol{\Psi}_3((1 - \hat{\rho}_\alpha(\theta))\hat{\mu}(\theta))$ (note that the total variation norm of the measure $(\delta_0 - \rho_\alpha) \star \mu$ might exceed the value 1, which means that a more direct reasoning by an obvious extension of (1·1) to signed measures in the unit ball of $(\mathbb{M}, \|\cdot\|_{\mathrm{TV}})$ does not work). Obviously, $\boldsymbol{\Sigma}(\delta_0 - \rho_{2\alpha}) \in \mathbb{M}(\gamma)$, so using Lemma 5(iii) and (iv) we see that ϕ_1 is the Fourier transform of a measure with tail decrease $O(x^{-\gamma})$. By inspection,

$$\inf_{|\theta| \leqslant 8\alpha} \big|\hat{\mu}_0(\theta)\big| > 0, \qquad \sup_{|\theta| \geqslant \alpha} \big|\hat{\mu}_0(\theta)\big| < 1,$$

so we can argue similarly with μ_0 in place of μ, which means that ϕ_2 is also the Fourier transform of a measure with tail decrease $O(x^{-\gamma})$.

In order to analyse ϕ_3 we first note that the factors $1 - \hat{\rho}_{2\alpha}(\theta)$ in $\phi_1(\theta)$ and $\phi_2(\theta)$ made it possible to change $\hat{\mu}(\theta)$ in a neighbourhood of $\theta = 0$ and that we can now similarly alter $\hat{\mu}(\theta)$ on $|\theta| \geqslant 4\alpha$. We require some further auxiliary measures. First, let $\mu_1 \coloneqq \boldsymbol{\Sigma}\mu + \delta_0 - \mu$; obviously, $\mu_1 \in \mathbb{M}(\gamma)$. Lemma 5(ii) shows that

$$\hat{\mu}_1(\theta) = \big(1 - \hat{\mu}(\theta)\big)\left(1 - \frac{1}{i\theta}\right) \quad \text{for } \theta \neq 0, \qquad \hat{\mu}_1(0) = 1.$$

Considering the real parts of the factors separately for $\theta \neq 0$ as in [**12**] we see that the range of $\hat{\mu}_1$ is contained in U_2. If a maximal ideal I contains $\mathbb{M}_a$ then the same arguments as used above for ϕ_1 yield $\tilde{\mu}_1(I) \in U_2$, hence we can apply Proposition 3 with $(U_2, \boldsymbol{\Psi}_2)$ to conclude that $\log \hat{\mu}_1(\theta)$ is the Fourier transform of an element of $\mathbb{M}(\gamma)$. Further, let $\mu_2 \coloneqq \mu + \delta_0 - \rho_{4\alpha}$; obviously, μ_2 and $\boldsymbol{\Sigma}\mu_2$ are in $\mathbb{M}(\gamma)$. For $|\theta| \geqslant 8\alpha$ we have $\hat{\rho}_{4\alpha}(\theta) = 0$ so that $\hat{\mu}_2(\theta) = \hat{\mu}(\theta) + 1 \neq 0$ because of (2·7). For $|\theta| < 8\alpha$ we obtain on using (2·6)

$$\mathrm{Re}\big(\hat{\mu}_2(\theta)\big) \geqslant \inf_{|\theta| \leqslant 8\alpha} (\mathrm{Re}(\hat{\mu}(\theta)) + 1 - \hat{\rho}_{4\alpha}(\theta)) > 0,$$

hence $\hat{\mu}_2(\theta) \neq 0$. If I is a maximal ideal containing $\mathbb{M}_a$ then we get, as above,

$$\tilde{\mu}_2(I) = \tilde{\mu}(I) + 1 \neq 0$$

because of $\big|\tilde{\mu}(I)\big| < 1$. Hence a similar reasoning as used above for the logarithm of $\hat{\mu}_1(\theta)$, now with $(U_1, \boldsymbol{\Psi}_1)$ instead of $(U_2, \boldsymbol{\Psi}_2)$, shows that there exists a $\mu_3 \in \mathbb{M}(\gamma)$ with

$$\hat{\mu}_3(\theta) = \frac{1}{\hat{\mu}(\theta) + 1 - \hat{\rho}_{4\alpha}(\theta)} \quad \text{for all } \theta \in \mathbb{R}. \tag{2·9}$$

We claim that $\boldsymbol{\Sigma}\mu_3 \in \mathbb{M}(\gamma)$. To see this we note that the associated Fourier transform can be written as

$$\widehat{\boldsymbol{\Sigma}\mu_3}(\theta) = -\hat{\mu}_3(\theta)\,\widehat{\boldsymbol{\Sigma}\mu_2}(\theta),$$

which shows that $\boldsymbol{\Sigma}\mu_3$ is the convolution product of two elements of $\mathbb{M}(\gamma)$.

Using μ_3 and $\boldsymbol{\Sigma}\mu_3$ we can now rewrite the Fourier transform of the last term in the decomposition (2·8) as follows,

$$\phi_3(\theta) = \hat{\rho}_{2\alpha}(\theta)\,(\hat{\mu}_3(\theta) - 1)\log\hat{\mu}_1(\theta) - \hat{\rho}_{2\alpha}(\theta)\,\big(\widehat{\Sigma\mu_3}(\theta) + 1\big)\, i\theta \log\left(1 - \frac{1}{i\theta}\right).$$

From the above reasoning we obtain that the factors in the first product on the right hand side are transforms of elements of $\mathbb{M}(\gamma)$. Note that the factor $\mu_3 - \delta_0$ has total mass 0 and that Σ, applied to this difference, yields an element of $\mathbb{M}(\gamma)$. Hence we obtain $O(x^{-\gamma})$-behaviour for the tails associated with this first product on using Lemma 5(iii) and (iv) again.

Overall, it remains to establish a suitable expansion for the measure μ_4 with Fourier transform

$$\hat{\mu}_4(\theta) = \hat{\rho}_{2\alpha}(\theta)\,(\widehat{\Sigma\mu_3}(\theta) + 1)\, i\theta\log\left(1 - \frac{1}{i\theta}\right).$$

Combining Lemmas 8 and 9 (with $\Sigma\mu_3$ for μ) we obtain

$$\hat{\mu}_4(\theta) = \hat{\rho}_{2\alpha}(\theta)\,(\hat{\mu}_5(\theta) + \hat{\mu}_6(\theta))$$

with $\mu_5((-\infty, -x]) = O(x^{-\gamma}\log x)$, $\mu_5((x, \infty)) = O(x^{-\gamma}\log x)$ as $x \to \infty$ and

$$\mu_6 := \sum_{j=1}^{\lceil\gamma\rceil - 2} (\Sigma^{j+1}\mu_3)(\mathbb{R})\left(\nu_{j+1} - \frac{1}{j+1}\,\delta_0\right).$$

The last statement in Lemma 4 yields $\rho_{2\alpha} \star \mu_5(I(x)) = O(x^{-\gamma}\log x)$ for $I(x) = (-\infty, -x]$ and $I(x) = (x, \infty)$. Lemma 7 implies that the tails of $\rho_{2\alpha} \star \mu_6$ differ from those of μ_6 by a negligible amount.

Putting all these pieces together we see that

$$(\nu_{\mathrm{rr}}(\mu) - \nu_{\mathrm{rr}}(\mu_0) + \mu_6)((x, \infty)) = O(x^{-\gamma}\log x), \tag{2·10}$$

and, as both $\nu_{\mathrm{rr}}(\mu_0)$ and μ_6 are concentrated on the right halfline,

$$\nu_{\mathrm{rr}}(\mu)((-\infty, -x]) = O(x^{-\gamma}\log x)$$

as $x \to \infty$. The latter implies the statement of the theorem for the left tail, hence it remains to check the behaviour of the (right) tail of μ_6. Lemma 5(v) yields

$$(\Sigma^{j+1}\mu_3)(\mathbb{R}) = \frac{1}{(j+1)!}\, m_{j+1}(\mu_3).$$

As (2·9) implies that $\hat{\mu}_3(\theta)\hat{\mu}(\theta) = 1$ for all θ in some neighbourhood of 0 we have $m_j(\mu_3) = r_j(\mu)$ by the remarks following (2·1). Further, Lemma 6 yields

$$\nu_{j+1}((x, \infty)) = f_j(x) = j!\, x^{-j-1} + o(e^{-x}),$$

so we obtain

$$\mu_6((x, \infty)) = \sum_{j=1}^{\lceil\gamma\rceil - 2} \frac{r_{j+1}(\mu)}{j+1}\, x^{-j-1} + O(x^{-\gamma}\log x). \tag{2·11}$$

From (2·10) and (2·11) the right tail formula (1·7) of the theorem follows.

3. *Comments*

Some extensions and variations of our main result are obvious, for example to rate functions more general than $x^{-\gamma}$, others would require some more work. In particular, we would expect that the above methods also yield tail expansions in the case that the first moment of the input distribution vanishes (see [**12**] for the corresponding situation in harmonic renewal theory). With the proper smoothness conditions on μ it is also possible to obtain expansions for the densities rather than the tails of random record distributions. For the lattice case, where μ is concentrated on the multiples of some $h > 0$, this has been carried out in [**20**].

There is some similarity between our concept of inverse moments and the classical concept of cumulants. The latter play a special role in the context of the central limit theorem, where the distribution with vanishing remainder term is the normal distribution which has all cumulants equal to zero from order three onwards. Example 2 shows that this role is taken over by the exponential distribution in the random record context. Are exponential distributions characterized by the requirement that all inverse moments from order two onwards vanish? This condition implies that all (ordinary) moments exist and do not increase too rapidly, so, as the ordinary moments can be obtained from inverse moments, such a characterization does indeed hold.

In summary, the above shows that Gelfand theory together with some elementary, albeit lengthy arguments leads to tail expansions for yet another type of convolution series, namely $\sum_{n=1}^{\infty} n^{-1}(n+1)^{-1}\mu^{\star n}$. The techniques can obviously be applied to other series such as $\sum_{n=1}^{\infty} n^{-1}(n+1)^{-1}(n+2)^{-1}\mu^{\star n}$. What we do not have at the moment is an explanation for the qualitatively very different form of the expansions for these series as compared to the classical renewal and harmonic renewal case. Gelfand theory provides the connection between convolution series $\sum_{n=1}^{\infty} a_n\mu^{\star n}$ and the functions $\phi: U \subset \mathbb{C} \to \mathbb{C}$, $z \mapsto \sum_{n=1}^{\infty} a_n z^n$, and it seems natural to look at the nature of the singularity of ϕ at $z = 1$ for an explanation. However, the one observation we can offer at present is the fact that $1/\phi'(z)$ is a polynomial in the renewal and harmonic renewal case and not in the random record case. Whether this is an algebraic coincidence or whether such an observation might lead to a classification of convolution series with respect to the qualitative type of their tail expansions we do not know.

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